ON THE ACCURACY LIMIT OF JOINT TIME-DELAY/DOPPLER/ACCELERATION ESTIMATION WITH A BAND-LIMITED SIGNAL

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ABSTRACT

The derivation of estimation lower bounds is paramount to design and assess the performance of new estimators. A lot of effort has been devoted to the joint distance-velocity estimation problem, but very few works deal with acceleration, being a key aspect in several high-dynamics applications. Considering a generic band-limited signal formulation, in this contribution we derive a new closed-form Cramér-Rao bound (CRB) expression for joint time-delay/Doppler/acceleration estimation. This new formulation, especially easy to use, depends only on the baseband signal samples, and can be exploited for several purposes including estimator assessment (i.e., for signal design or to derive performance loss metrics with respect to the best (lowest) CRB). These results are illustrated and validated with two representative band-limited signals, namely, a GPS L1 C/A signal and a linear frequency modulated chirp signal.

Index Terms— Cramér-Rao bound, delay/Doppler/acceleration estimation, band-limited signals, maximum likelihood.

1. INTRODUCTION

Deterministic parameter estimation appears in a plethora of applications, such as radar and navigation [1–5], being a key stage of the receiver. Different estimation techniques exist and a common goal for estimator design is to assess the best achievable performance, and information which is brought by lower performance bounds [6]. In the large sample regime [7] or in the large signal-to-noise ratio (SNR) of the Gaussian conditional model [8, 23], accurate estimation of the minimum achievable mean squared error (MSE) is given by deterministic Cramér-Rao Bounds (CRB). Several time-delay and Doppler estimation CRB expressions have been already derived for different applications [9–13]. In addition, easy-to-use delay/Doppler closed-form CRB expressions, which only depend on the baseband signal samples, have been recently derived and exploited in [14–17]. However, even if acceleration plays an important role in several high-dynamics applications, a compact CRB in the vein of [14] including acceleration estimation in addition to time-delay and Doppler, for a generic band-limited signal, is not available in the literature.

Indeed, in certain applications, adding acceleration estimation in the first receiver stage may improve the overall estimation performance (i.e., notice that the assumption of constant Doppler during the observation time was done in the contributions hereabove). On the other hand, because an additional parameter needs to be estimated, the resulting bounds are expected to be higher than the ones found in [14] for joint delay and Doppler estimation. In this contribution, we provide a new closed-form CRB expression for the joint delay/Doppler/acceleration estimation problem, for generic band-limited signals with given length and energy, of broad interest for several applications. Such CRBs are validated in two different ways: i) analysing the correct fit of the ambiguity function with its second order Taylor expansion, and ii) the convergence of the Maximum Likelihood Estimator (MLE) to the CRB at high SNR. Notice that because the proposed expressions depend only on the baseband signal samples, the CRBs can be easily implemented, therefore being a fundamental tool to compare the performance brought by different signals (i.e., as done in [15, 16] for Global Navigation Satellite Systems (GNSS) signals), as well as for signal design. These new CRB results are illustrated and validated with two representative band-limited signals, namely, a GPS L1 C/A signal and a linear frequency modulated chirp signal.

2. SIGNAL MODEL

To define the estimation problem of interest, consider the line-of-sight transmission of a band-limited signal $s(t)$ with sampling frequency F_s over a carrier with frequency f_c (wavelength $\lambda_c = c/f_c$), from a transmitter T at position $\mathbf{p}_T\left(t\right)$ to a receiver R at position $\mathbf{p}_R\left(t\right)$, expressed both in time and frequency as,

$$s\left(t\right) = \sum_{n=-N_1'}^{N_2'} s\left(\frac{n}{F_s}\right) \operatorname{sinc}\left(\pi F_s \left(t - \frac{n}{F_s}\right)\right), \tag{1a}$$

$$s\left(f\right) = \frac{1}{F_s} \sum_{n=-N_1'}^{N_2'} s\left(\frac{n}{F_s}\right) e^{-j2\pi n \frac{f}{F_s}},\ \frac{-F_s}{2} \leq f \leq \frac{F_s}{2}. \tag{1b}$$

The received signal at the output of the receiver antenna in the time domain is $x_A(t) = \alpha_A e^{j2\pi f_c (t-\tau_0(t;\boldsymbol{\eta}))} s\left(t - \tau_0(t;\boldsymbol{\eta})\right) + n_A(t)$, with $\boldsymbol{\eta} = \left[\tau, b, d\right]^T$ the unknown parameters to be estimated (i.e., related to the delay, Doppler and acceleration, respectively), $n_A(t)$ a zero-mean white complex circular Gaussian noise, and α_A an amplitude factor that depends on signal power, polarisation vectors, antenna gains, and antenna [18, 19].

The radial displacement between transmitter and receiver $(\mathbf{p}_{TR})$ is proportional to the transmitter to receiver signal time-delay (i.e., $\mathbf{p}_T(t) = \mathbf{p}_T + \mathbf{v}_T t + \frac{1}{2}\mathbf{a}_T t^2$ and $\mathbf{p}_R(t) = \mathbf{p}_R + \mathbf{v}_R t + \frac{1}{2}\mathbf{a}_R t^2$). However, notice that the perceived signal delay at the receiver is affected by the relative motion between both transmitter and receiver (i.e., Doppler effect). To correctly represent the shifted signal delay for high-dynamic receivers, in addition of the Doppler effect, we take into account the relative acceleration. The distance between two bodies with a relative acceleration depends

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on time, velocity, and acceleration and is obtained from integrating the acceleration with respect to time twice. The equation which describes the distance travelled by the transmitted signal is then $\mathbf{p}_{TR} \triangleq \|\mathbf{p}_T(t-\tau_0(t)) - \mathbf{p}_R(t)\| = c\tau_0(t) \simeq (\mathbf{p}_T - \mathbf{p}_R) + vt + \frac{at^2}{2}$, that is, a second order approximation. Then, $\tau_0(t) \simeq \tau + bt + dt^2, \tau = \frac{\mathbf{p}_T - \mathbf{p}_R}{c}, b = \frac{v}{c}, d = \frac{a}{2c}$ and c the speed of light. For simplification on the calculations required for the CRB, the signal model is considered to be narrowband, that is, the impact of the parameters linked to the Doppler shift b and the acceleration d is negligible in the received baseband signal, $s(t-\tau_0(t)) \simeq s(t-\tau)$. For short observation times, a good approximation of the baseband output of the receiver's Hilbert filter is [20],

$$a(t;\boldsymbol{\eta}) = e^{-j2\pi f_c(b(t-\tau)+d(t-\tau)^2)} s(t-\tau), \tag{2}$$

$$x(t) = x_A(t)\, e^{-j2\pi f_c t} = \alpha a(t;\boldsymbol{\eta}) + n(t), \tag{3}$$

with $f \in \left[-\frac{F_s}{2}, \frac{F_s}{2}\right]$, $F_s \geq B$ the Hilbert filter bandwidth, $n(t)$ a complex white circular Gaussian noise within this bandwidth with unknown variance σ_n^2, and $\alpha = \alpha_A e^{-j2\pi f_c \tau}$. The discrete vector signal model is built from $N = N_1 + N_2 + 1$ ($N_1/F_s \gg N_1'/B$, $N_2/F_s \gg N_2' B$) samples at $T_s = 1/F_s$,

$$\mathbf{x} = \alpha \mathbf{a}(\boldsymbol{\eta}) + \mathbf{n} = \rho e^{j\varphi} \mathbf{a}(\boldsymbol{\eta}) + \mathbf{n}, \tag{4}$$

with (for $N_1 \leq k \leq N_2$) signal samples $\mathbf{x} = (\ldots, x(kT_s), \ldots)^\top$, noise samples $\mathbf{n} = (\ldots, n(kT_s), \ldots)^\top$, baseband signal samples $\mathbf{s}(\tau) = (\ldots, s(kT_s - \tau), \ldots)^\top$ and from (2) we have that $\mathbf{a}(\boldsymbol{\eta}) = (\ldots, s(kT_s - \tau)e^{-j2\pi f_c(b(kT_s-\tau)+d(kT_s-\tau)^2)}, \ldots)^\top$. The unknown deterministic parameters can be gathered in vector $\boldsymbol{\epsilon} = \left(\sigma_n^2, \rho, \varphi, \tau, b, d\right)^\top$, with $\alpha = \rho e^{j\varphi}$ ($\rho \in \mathbb{R}^+$, $0 \leq \varphi \leq 2\pi$). In the sequel we focus on the CRB associated to the estimation of $\boldsymbol{\eta}$ because they are the main parameters of interest. In this contribution a single line-of-sight transmission is considered, however, (4) can also be obtained if one considers transmission via diffraction, reflexion or a combination of the three (multipaths) [2, 19].

3. BACKGROUND

Let $S = span(\mathbf{A})$ be the linear span of the set of the column vectors of matrix $\mathbf{A}$, $\mathbf{\Pi_A} = \mathbf{A}\left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H$ is the orthogonal projection over S, and $\mathbf{\Pi}_\mathbf{A}^\perp = \mathbf{I} - \mathbf{\Pi_A}$. The CRB is given by the inverse of the Fisher Information Matrix (FIM), $\mathbf{F}_{[\tau,b,d]|\boldsymbol{\epsilon}}\left(\boldsymbol{\epsilon}^0\right)$, with $\boldsymbol{\epsilon}^0$ a selected value of $\boldsymbol{\epsilon}$. If we define,

$$\Phi(\boldsymbol{\eta}) = \frac{\partial \mathbf{a}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^T}^H \mathbf{\Pi}_{\mathbf{a}(\boldsymbol{\eta})}^\perp \frac{\partial \mathbf{a}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^T} = \left\|\frac{\partial \mathbf{a}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^T}\right\|^2 - \frac{\left|\mathbf{a}(\boldsymbol{\eta})^H \frac{\partial \mathbf{a}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^T}\right|^2}{\|\mathbf{a}(\boldsymbol{\eta})\|^2}, \tag{5}$$

the FIM associated to the problem of interest is given by [21, 22]

$$\mathbf{F}_{\boldsymbol{\eta}|\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}) = \frac{2|\alpha|^2}{\sigma_n^2} \Re\{\boldsymbol{\Phi}(\boldsymbol{\eta})\} = \begin{bmatrix} F_{\tau|\boldsymbol{\epsilon}} & F_{\tau,b|\boldsymbol{\epsilon}} & F_{\tau,d|\boldsymbol{\epsilon}} \\ F_{b,\tau|\boldsymbol{\epsilon}} & F_{b|\boldsymbol{\epsilon}} & F_{b,d|\boldsymbol{\epsilon}} \\ F_{d,\tau|\boldsymbol{\epsilon}} & F_{d,b|\boldsymbol{\epsilon}} & F_{d|\boldsymbol{\epsilon}} \end{bmatrix}, \tag{6}$$

and the CRB matrix can be computed as $\mathbf{CRB}_{\boldsymbol{\eta}|\boldsymbol{\epsilon}} = \mathbf{F}_{\boldsymbol{\eta}|\boldsymbol{\epsilon}}^{-1}$. Moreover, (6) can be directly obtained through the following formula

$$\begin{aligned}\Re\{\boldsymbol{\Phi}(\boldsymbol{\eta})\} &= \Re\left\{\frac{\partial \mathbf{a}(\boldsymbol{\eta})^H}{\partial \boldsymbol{\eta}^T} \frac{\partial \mathbf{a}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^T}\right\} \\ &- \Re\left\{-\frac{1}{\|\mathbf{a}(\boldsymbol{\eta})\|^2}\left(\mathbf{a}(\boldsymbol{\eta})^H \frac{\partial \mathbf{a}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^T}\right)^H \left(\mathbf{a}(\boldsymbol{\eta})^H \frac{\partial \mathbf{a}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^T}\right)\right\}, \end{aligned} \tag{7}$$

where the partial derivative vector can be expressed in terms of the following matrices:

$$\frac{\partial \mathbf{a}(t;\boldsymbol{\eta})}{\partial \boldsymbol{\eta}} = -\mathbf{Q}\boldsymbol{\vartheta} e^{-j\omega_c(b(t-\tau)+d(t-\tau)^2)}, \tag{8}$$

with

$$\mathbf{Q} = \begin{bmatrix} -j\omega_c b & -j2\omega d & 1 & 0 \\ 0 & j\omega_c & 0 & 0 \\ 0 & 0 & 0 & j\omega_c \end{bmatrix}, \boldsymbol{\vartheta} = \begin{bmatrix} s(t;\boldsymbol{\eta}) \\ (t-\tau)s(t;\boldsymbol{\eta}) \\ s^{(1)}(t;\boldsymbol{\eta}) \\ (t-\tau)^2 s(t;\boldsymbol{\eta}) \end{bmatrix}, \tag{9}$$

and $s^{(1)}(t) = \frac{ds(t)}{dt}$. Representing the signal with discrete time values allows the products to be expressed as sums,

$$\begin{aligned} &\|\mathbf{a}(\boldsymbol{\eta})\|^2 = \textstyle\sum_{n=N_1'}^{N_2'} |s(nT_s;\boldsymbol{\eta})|^2, \\ &\mathbf{a}(\boldsymbol{\eta})^H \frac{\partial \mathbf{a}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^T} = -\left(\textstyle\sum_{n=N_1'}^{N_2'} \boldsymbol{\vartheta}(nT_s;\boldsymbol{\eta}) s^*(nT_s;\boldsymbol{\eta})\right)^T \mathbf{Q}^T, \\ &\frac{\partial \mathbf{a}(\boldsymbol{\eta})^H}{\partial \boldsymbol{\eta}^T} \frac{\partial \mathbf{a}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}^T} = \mathbf{Q}^* \left(\textstyle\sum_{n=N_1'}^{N_2'} \boldsymbol{\vartheta}^*(nT_s;\boldsymbol{\eta}) \boldsymbol{\vartheta}^T(nT_s;\boldsymbol{\eta})\right) \mathbf{Q}^T. \end{aligned}$$

3.1. Maximum Likelihood Estimation and Ambiguity Function

Considering the signal model (4), the MLE is defined as [22],

$$\widehat{\boldsymbol{\eta}} = \arg\min_{\boldsymbol{\eta}} \left\{\mathbf{x}^H \mathbf{\Pi}_{\mathbf{a}(\boldsymbol{\eta})}^\perp \mathbf{x}\right\} = \arg\max_{\boldsymbol{\eta}} \left\{\frac{\left|\mathbf{a}(\boldsymbol{\eta})^H \mathbf{x}\right|^2}{\mathbf{a}(\boldsymbol{\eta})^H \mathbf{a}(\boldsymbol{\eta})}\right\} \tag{10}$$

$$\underset{\min(N_1,N_2)\to\infty}{=} \arg\max_{\boldsymbol{\eta}} \left\{\frac{\left|\int_{-\infty}^{+\infty} \mathbf{a}(t;\boldsymbol{\eta})^* x(t)\, dt\right|^2}{\int_{-\infty}^{+\infty} |s(t)|^2\, dt}\right\}, \tag{11}$$

and the maximum SNR at the output of the MLE matched filter is,

$$\mathrm{SNR}_{\mathrm{out}} = \frac{\left|\int_{-\infty}^{+\infty} \mathbf{a}(t;\boldsymbol{\eta})^* \alpha \mathbf{a}(t;\boldsymbol{\eta}) dt\right|^2}{E\left[\left|\int_{-\infty}^{+\infty} \mathbf{a}(t;\boldsymbol{\eta})^* n(t)\, dt\right|^2\right]} \tag{12}$$

$$= \frac{|\alpha|^2 \left(\int_{-\infty}^{+\infty} |s(t)|^2\, dt\right)^2}{\left(\frac{\sigma_\mathbf{n}^2}{F_s}\right) \int_{-\frac{B}{2}}^{\frac{B}{2}} |s(f)|^2\, df} = \frac{|\alpha|^2\, \mathbb{E}}{\left(\frac{\sigma_\mathbf{n}^2}{F_s}\right)}. \tag{13}$$

Last, the corresponding ambiguity function is given by [19],

$$\Xi\left(\boldsymbol{\eta};\boldsymbol{\eta}^0\right) = \left|\frac{\mathbf{a}(\boldsymbol{\eta})^H \mathbf{a}\left(\boldsymbol{\eta}^0\right)}{\|\mathbf{a}(\boldsymbol{\eta})\| \left\|\mathbf{a}\left(\boldsymbol{\eta}^0\right)\right\|}\right|^2, \tag{14}$$

which can be approximated by its 2nd order Taylor expansion as,

$$\Xi\left(\boldsymbol{\eta}^0 + d\boldsymbol{\eta};\boldsymbol{\eta}^0\right) \simeq 1 - \frac{1}{2} d\boldsymbol{\eta}^T \left(\frac{2\Re\left\{\Phi\left(\boldsymbol{\eta}^0\right)\right\}}{\left\|\mathbf{a}\left(\boldsymbol{\eta}^0\right)\right\|^2}\right) d\boldsymbol{\eta}, \tag{15}$$

where the second term is directly related to the FIM (6), and therefore to the CRB.

4. NEW CLOSED-FORM JOINT TIME-DELAY, DOPPLER AND ACCELERATION ESTIMATION CRB

Taking $s(t)$ as a band-limited signal, and using the Nyquist-Shannon theorem, we have that [17],

$$\lim_{N_1 \to -\infty, N_2 \to \infty} \Re\left\{\boldsymbol{\Phi}(\boldsymbol{\eta})\right\} = F_s \Re\left\{\mathbf{Q}\mathbf{W}\mathbf{Q}^H - \frac{(\mathbf{Q}\mathbf{w})(\mathbf{Q}\mathbf{w})^H}{w_1}\right\}, \tag{16}$$

with $\mathbf{w} = [w_1, w_2, w_3, w_4]^\top$ and

$$\mathbf{W} = \begin{bmatrix} w_1 & w_2^* & w_3^* & w_4^* \\ w_2 & W_{2,2} & W_{3,2}^* & W_{4,2}^* \\ w_3 & W_{3,2} & W_{3,3} & W_{4,3}^* \\ w_4 & W_{4,2} & W_{4,3} & W_{4,4} \end{bmatrix}$$

with

$$w_1 = \int_{-\infty}^{\infty} |s(t;\boldsymbol{\eta})|^2 \, dt, \;\; w_2 = \int_{-\infty}^{\infty} (t-\tau) \, |s(t;\boldsymbol{\eta})|^2 \, dt,$$

$$w_3 = \int_{-\infty}^{\infty} s^{(1)}(t;\boldsymbol{\eta}) s(t;\boldsymbol{\eta}) dt, \;\; W_{3,3} = \int_{-\infty}^{\infty} \left|s^{(1)}(t;\boldsymbol{\eta})\right|^2 dt,$$

$$w_4 = W_{2,2} = \int_{-\infty}^{\infty} (t-\tau)^2 \, |s^*(t;\boldsymbol{\eta})|^2 \, dt,$$

$$W_{3,2} = \int_{-\infty}^{\infty} (t-\tau) s^{(1)}(t;\boldsymbol{\eta}) s^*(t;\boldsymbol{\eta}) dt,$$

$$W_{4,3} = \int_{-\infty}^{\infty} (t-\tau)^2 s^{(1)}(t;\boldsymbol{\eta}) s^*(t;\boldsymbol{\eta}) dt.$$

The previous analytic expressions were already computed as a function of the signals samples in [17]. To obtain the missing closed-form expressions of $W_{4,2}$ and $W_{4,4}$, the integral equations are evaluated with the use of Fourier transform properties: $s(t) \rightleftharpoons s(f)$, $ts(t) \rightleftharpoons \frac{j}{2\pi}\frac{ds(f)}{df}$, $t^2 s(t) \rightleftharpoons -\frac{1}{4\pi^2}\frac{d^2 s(f)}{df^2}$, and then

$$W_{4,2} = \int_{-\infty}^{\infty} (t-\tau)^3 \, |s^*(t;\boldsymbol{\eta})|^2 \, dt = \int_{-\infty}^{\infty} (t^2 s(t))(ts(t))^* dt$$
$$\rightleftharpoons \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} -\frac{1}{4\pi^2}\frac{d^2 s(f)}{df^2}\left(\frac{j}{2\pi}\frac{ds(f)}{df}\right)^* df = \frac{1}{F_s^4}\mathbf{s}^H \mathbf{D}^3 \mathbf{s}, \tag{17}$$

$$W_{4,4} = \int_{-\infty}^{\infty} (t-\tau)^4 \, |s^*(t;\boldsymbol{\eta})|^2 \, dt = \int_{-\infty}^{\infty} (t^2 s(t))(t^2 s(t))^* dt$$
$$\rightleftharpoons \int_{-\frac{F_s}{2}}^{\frac{F_s}{2}} -\frac{1}{4\pi^2}\frac{d^2 s(f)}{df^2}\left(-\frac{1}{4\pi^2}\frac{d^2 s(f)}{df^2}\right)^* df = \frac{1}{F_s^5}\mathbf{s}^H \mathbf{D}^4 \mathbf{s} \tag{18}$$

where $\mathbf{D} = diag([N_1, N_1+1, ..., N_2-1, N_2])$ and

$$\frac{ds(f)}{df} = \frac{-j2\pi}{F_s^2}\sum_{n=N_1}^{N_2} n s(nT_s) e^{-j2\pi\frac{f}{F_s}n}, \tag{19}$$

$$\frac{d^2 s(f)}{df^2} = \frac{-4\pi^2}{F_s^3}\sum_{n=N_1}^{N_2} n^2 s(nT_s) e^{-j2\pi\frac{f}{F_s}n}. \tag{20}$$

Finally, after tedious calculus, the terms in (16) are obtained by taking the limit, as shown in the following for the first term,

$$\lim_{\substack{N1 \to -\infty \\ N2 \to \infty}} \Re\left\{\Phi_{1,1}(\boldsymbol{\eta})\right\} = F_s \begin{bmatrix} W_{3,3} - \frac{|w_3|^2}{w_1} \\ +4\omega_c^2 d^2 \left(W_{2,2} - \frac{|w_2|^2}{w_1}\right) \\ -4\omega_c d \Im\left\{W_{3,2} - \frac{w_2^* w_3}{w_1}\right\} \end{bmatrix}. \tag{21a}$$

The same limit must be taken for all following equations but is omitted for simplicity,

$$\Re\left\{\Phi_{2,2}(\boldsymbol{\eta})\right\} = F_s\left(\omega_c^2\left(W_{2,2} - \frac{|w_2|^2}{w_1}\right)\right), \tag{21b}$$

$$\Re\left\{\Phi_{3,3}(\boldsymbol{\eta})\right\} = F_s\left(\omega_c^2\left(W_{4,4} - \frac{|w_4|^2}{w_1}\right)\right), \tag{21c}$$

$$\Re\left\{\Phi_{1,2}(\boldsymbol{\eta})\right\} = F_s\begin{pmatrix} 2\omega_c^2 d\left(\frac{|w_2|^2}{w_1} - W_{2,2}\right) \\ -\omega_c \Im\left\{\frac{w_2^* w_3}{w_1} - W_{3,2}\right\}\end{pmatrix}$$
$$= \Re\left\{\Phi_{2,1}(\boldsymbol{\eta})^*\right\}, \tag{21d}$$

$$\Re\left\{\Phi_{1,3}(\boldsymbol{\eta})\right\} = F_s\begin{pmatrix} 2\omega_c^2 d\left(\frac{w_2 w_4^*}{w_1} - W_{4,2}^*\right) \\ -\omega_c \Im\left\{\frac{w_4^* w_3}{w_1} - W_{4,3}^*\right\}\end{pmatrix}$$
$$= \Re\left\{\Phi_{3,1}(\boldsymbol{\eta})^*\right\}, \tag{21e}$$

$$\Re\left\{\Phi_{2,3}(\boldsymbol{\eta})\right\} = F_s\left(\omega_c^2\left(W_{4,2}^* - \frac{w_2 w_4^*}{w_1}\right)\right) = \Re\left\{\Phi_{3,2}(\boldsymbol{\eta})^*\right\}. \tag{21f}$$

Finally, the CRB matrix can be computed by inverting the FIM, and the corresponding CRB values for the time-delay, Doppler and acceleration can be identified from the diagonal values,

$$\mathbf{CRB}_{\boldsymbol{\eta}|\boldsymbol{\epsilon}} = \frac{\sigma_n^2}{2|\alpha|^2}\Re\left\{\boldsymbol{\Phi}(\boldsymbol{\eta})\right\}^{-1} \tag{22}$$

It is worth noting that (22) does not depend on delay or Doppler (τ and b), but it does depend on the acceleration parameter (d). For higher acceleration values, the CRB cannot neglect this term when assessing the asymptotic precision of estimation of τ and b, which now depends on d. This is also why d must be accounted for in the MLE matched filter ((11) and (12)) when acceleration increases.

5. VALIDATION AND DISCUSSION

In order to assess the validity of the new CRB expressions we consider two representative band-limited signals: i) a GPS L1 C/A signal with PRN code length 1023 chips, $F_s = 1.023$ MHz, and a duration of 10 ms (i.e., 10 consecutive PRN), and ii) a Linear Frequency Modulated (LFM) chirp signal, with a bandwidth equal to half the sampling frequency ($F_s = 1.023$ MHz) and duration 10 ms.

First, we want to assess if the ambiguity function and its 2nd order Taylor approximation coincide, which is a proof of exactness of the proposed CRBs. In that perspective we consider the GPS L1 C/A signal, and a scenario where only the acceleration parameter d is to be estimated (i.e., known delay and Doppler). In this case, the signal model simplifies to $a(t;\boldsymbol{\eta}) = s(t)e^{-j\omega_c dt^2}$, and the FIM can be directly computed using (21c). This result is illustrated in Figure 1, with a clear good fit between the ambiguity function and its approximation for a wide range of acceleration values around the true one, which is given by the maximum of the ambiguity function. This implies that the terms of the FIM in (21c) (e.g., $W_{4,4}$) are correct. This is a first proof of exactness of the CRB, but the complete estimation problem still needs to be characterized.

Since (4) belongs to the class of conditional signal models [23], the MLE converges to the CRB at high SNR [8], a property which can be used as an alternative way to confirms the CRB validity. Then, a second experiment with the signal model in (2) (i.e., τ, b and d

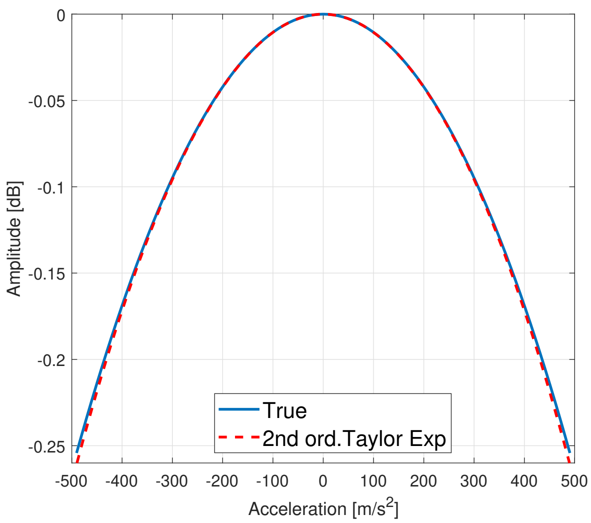

Fig. 1: Ambiguity function and second order Taylor approximation for a GPS L1 C/A band-limited signal.

are unknown parameters that need to be estimated), and for the two band-limited signals of interest, is carried out. The CRB and the corresponding MLE (10), obtained for 1000 Monte Carlo iterations and a normalized noise are shown in Figure 2. As expected, the MLE for the different parameters of interest converges to the corresponding CRB after a certain threshold region. The CRB and MLE behaviour in the different plots of Figure 2 confirm the exactness of the proposed CRB (21a)-(21f). Notice that the convergence threshold is slightly different for both signals. This threshold is equal to 17 dB for the GPS signal, and 15 dB for the LFM signal, in terms of the SNR at the output of the MLE matched filter. Apart from this slightly different asymptotic region of operation, the performance obtained with both signals is equivalent. In any case, both the good fit of the ambiguity function and the MSE of the MLE confirm the validity and exactness of the new closed-form CRB expressions for joint time-delay, Doppler and acceleration estimation.

6. CONCLUSION

In this contribution we derived novel CRB closed-form expressions for delay, Doppler and acceleration estimation. These expressions are valid for a generic narrowband band-limited signal, and therefore can be exploited in several applications such as navigation, radar, GNSS reflectometry or space exploration, to name a few. An appealing advantage is that the new CRB expressions only depend on the baseband signal samples, which make them especially easy to use. Results were illustrated using two representative band-limited signals, namely, a GPS L1 C/A signal and a LFM chirp signal. The validity of the new CRBs was demonstrated by comparing them with the corresponding MSE of the new MLE, as well as with the comparison of the ambiguity function and its second order Taylor approximation. Both results confirmed the validity and exactness of the proposed CRBs.

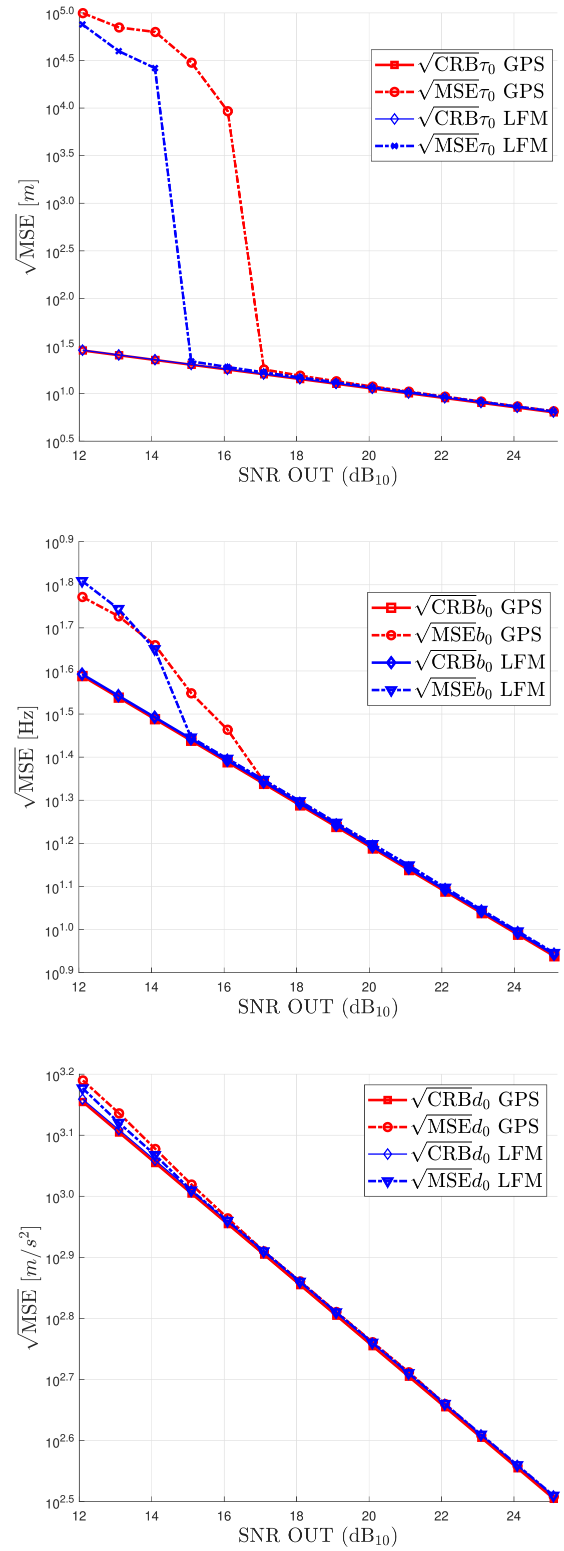

Fig. 2: CRB and MSE of delay (top), Doppler (middle) and acceleration (bottom), for a GPS L1 C/A and a LFM band-limited signals.

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