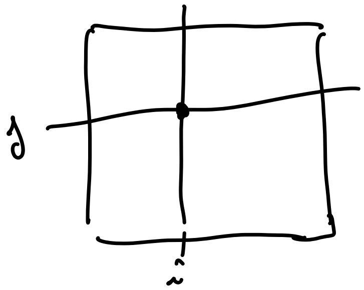
Cours 11/09/2023



$$\underbrace{z_{ij}}_{\in\{0,\dots,255\}} = v_{ij} + \underbrace{b_{ij}}_{X}$$

<u>Expérience aléatoire</u> : $(\Omega, \mathcal{T}, P)$

Ω : ensemble des résultats d'expérience

$\mathcal{T}$: ensemble des événements

P : application probabilité

Jet de dé $\Omega = \{1, \dots, 6\}$

$\mathcal{T} = \{ \{1\}, \dots, \{6\}, \{1,2\} \dots \{5,6\}, \dots, \Omega, \emptyset \}$

<u>Équiprobabilité</u> :
- Ω fini
- tous les événements élémentaires, i.e., les singletons $\{\omega\}$ ont la même importance

$$P(A) = \frac{\operatorname{Card} A}{\operatorname{Card} \Omega}$$

<u>Loi Binomiale</u> : n expériences de type {succès, échec}

$$P(k \text{ succès sur } n \text{ expériences}) = \underbrace{\binom{n}{k}}_{\frac{n!}{k!(n-k)!}} P_s^k (1-P_s)^{n-k} \qquad k \in \{0, \dots, n\}$$

$$C_n^k$$

P_s = probabilité du succès sur 1 expérience

<u>Jet de dé</u> : 10 jets, $n = 10$

$$P(3 \text{ fois } 2) = \binom{10}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7 \qquad k = 3$$

Proba conditionnelle

$$P(A \mid B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{\mathrm{Card}(A\cap B)}{\mathrm{Card}\,\Omega}}{\frac{\mathrm{Card}\,B}{\mathrm{Card}\,\Omega}}$$

sachant

équiprobabilité

$$\boxed{P(A\cap B) = P(A\mid B)\,P(B)}$$

A et B indépendants $\iff$ $\boxed{P(A\cap B) = P(A)\,P(B)}$

$P(A)\neq 0$

$P(B)\neq 0$

$\iff$ $\boxed{P(A\mid B) = P(A)}$

BAYES

$$\boxed{P(A\mid B) = \frac{P(B\mid A)\,P(A)}{P(B)}}$$

$P(A\mid B) = \dfrac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B) = P(A\mid B)\,P(B)$

$=$

$P(B\mid A) = \dfrac{P(B\cap A)}{P(A)} \Rightarrow P(B\cap A) = P(B\mid A)\,P(A)$

~~Somme de~~ Variables aléatoires NOTATION

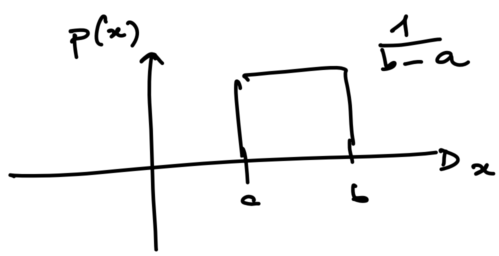
X : somme des résultats des 2 dés

$x = 4$ réalisation de X

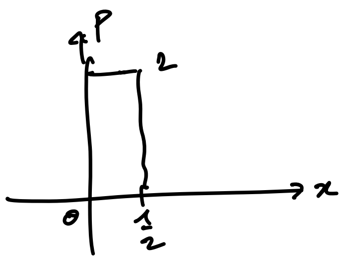
$P[X=4]$ probabilité que X prenne la valeur $x=4$

" $P[\{\omega / \underbrace{n+m=4}_{X(\omega)}\}]$

Loi uniforme sur $]a,b[$



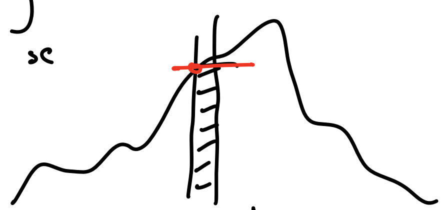
$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$$

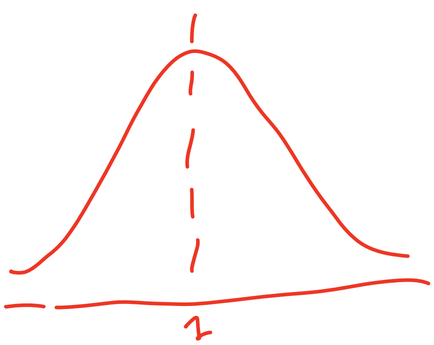


on peut avoir $p(x) > 1$ donc $p(x)$ n'est pas une probabilité

Que représente $p(x)$?

$$P[X \in \underbrace{]x, x+dx[}_{\Delta}] = \int_x^{x+dx} p(u)\,du$$

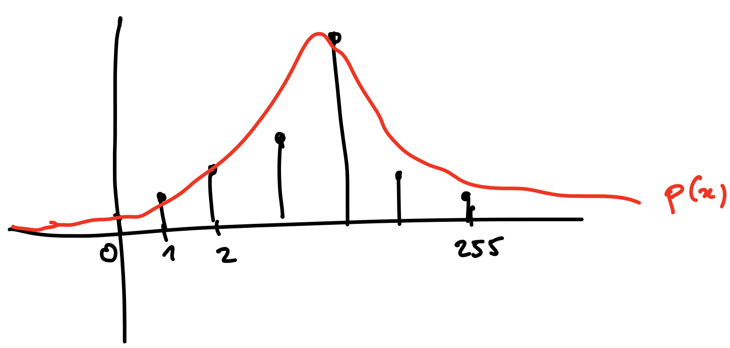


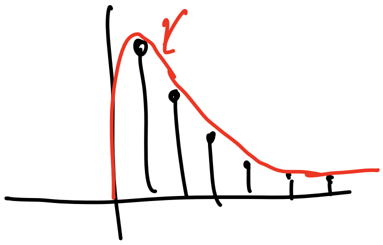


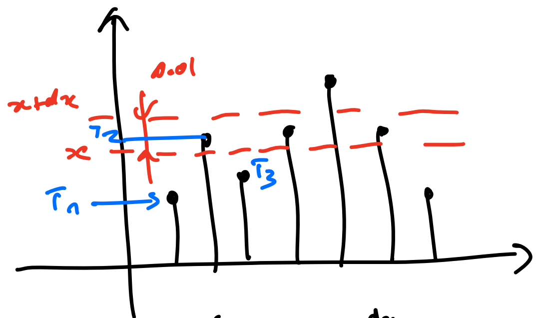
$$\underset{dx\ \text{"petit"}}{\approx} \int_x^{x+dx} p(u)\,du = p(x)\,dx$$

Conclusion

$$\boxed{p(x) \underset{dx\ \text{petit}}{\approx} \frac{P[X \in]x, x+dx[]}{dx}}$$



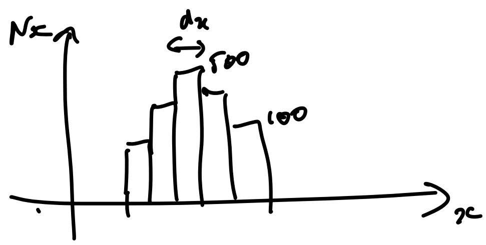




$$p(x)\,dx \approx P[\,X \in]x, x+dx[\,]$$

$$p(x) \approx \frac{P[\,X \in]x, x+dx[\,]}{dx}$$

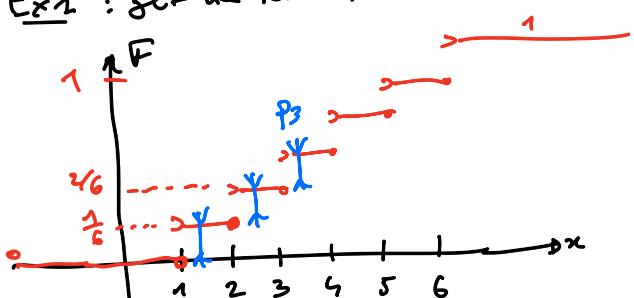
$$\approx \frac{N_x}{dx}$$ Histogramme des données.

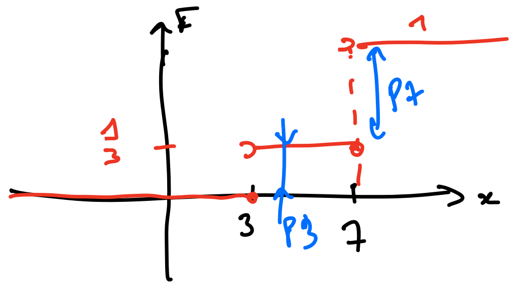


Fonctions de répartition

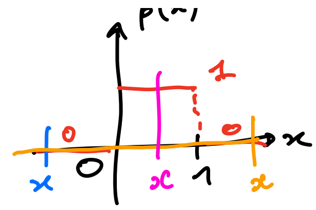
$$F(x) = P[\,X < x\,]$$

Ex 1 : jet de dé $\quad \Omega = \{1, \dots, 6\}$



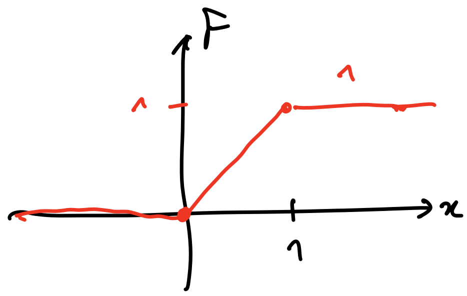


Loi uniforme sur]0,1[



$$F(x) = P[X < x] = P[X \in]-\infty, x[]$$

$$= \int_{-\infty}^{x} p(u)\,du = \begin{cases} 0 & \text{si } x \leq 0 \\ 1 & \text{si } x \geq 1 \\ \int_0^x 1\,du = x & x \in]0,1[\end{cases}$$



Rq : $F(x) = \int_{-\infty}^{x} p(u)\,du \Rightarrow \boxed{p(x) = F'(x)}$

MOYENNE $E[X] = \sum_i x_i p_i$

① Jet de dé $E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \dots + 6 \times \frac{1}{6}$

$$= \frac{1}{6}(1 + 2 + \dots + 6) = \frac{1}{6} \frac{6 \times 7}{2} = \boxed{3.5}$$

Remarque : si $x_1^*, x_2^*, \dots, x_n^*$ sont des réalisations d'une même variable aléatoire X, alors sous certaines hypothèses

Loi des grands nombres

$$\underbrace{\frac{x_1^* + x_2^* + \dots + x_n^*}{n}}_{\text{moyenne arithmétique}} \xrightarrow[n \to +\infty]{} E[X]$$

② Loi normale

$$\boxed{X \underset{\text{suit la loi}}{\sim} N(m,\sigma^2)}$$ NOTATION

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$$

$$E[X] = \int_{\mathbb{R}} x \, p(x) \, dx$$

$$= \int_{\mathbb{R}} x \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] dx}$$

$u = \dfrac{x-m}{\sigma}$

on centre et on réduit

$x = \sigma u + m$

$$= \int_{\mathbb{R}} (\sigma u + m) \underbrace{\frac{1}{\sqrt{2\pi\cancel{\sigma^2}}} \exp\left(-\frac{u^2}{2}\right)}_{\frac{1}{\sqrt{2\pi}} e^{-u^2/2}} \cancel{\sigma} \, \underbrace{du}_{dx}$$

$$= \int_{\mathbb{R}} (\sigma u + m) \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du$$

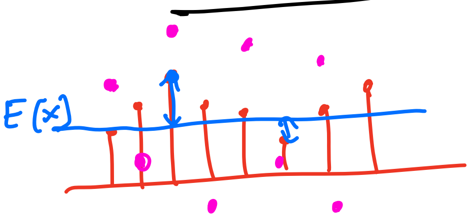
$$= \sigma \int_{\mathbb{R}} \underbrace{u \frac{1}{\sqrt{2\pi}} e^{-u^2/2}}_{\text{fonction impaire}} du + m \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du}_{1}$$

$= 0$

$$\boxed{E[X] = m}$$

Moments non centrés : $E(X)$, $E[X^2]$, $E[X^3]$, ...

Moments centrés : $\boxed{\underbrace{E[(X - E[X])]}}^{=0}$, $E[\underbrace{(X - E(X))^2}]$, ...



$$E(X) - E[\underbrace{E(X)}_{\text{cste}}] = 0$$

$\underbrace{\quad}_{E(X)}$

$(X - E(X))^2$

écart quadratique entre X et sa moyenne

↓

$\boxed{\text{VARIANCE}}$

Calcul de variance

$$\boxed{\sqrt{\text{variance}} = \text{écart type}}$$

Définition

$$\boxed{E[(X-E(X))^2]} = E[X^2 \underbrace{-2XE(X)}_{a} + \underbrace{E^2(X)}_{b}]$$

$$= E(X^2) - 2E(X)E(X) + E^2(X)$$

$$\boxed{\text{Var}\,X = E(X^2) - E^2(X)}$$ utilisé pour le calcul

Exemple de calcul

$X \sim N(\underset{E(X)}{m}, \sigma^2)$

$$\text{Var}\,X = E(X^2) - E^2(X) \qquad E(X) = m$$

$$E[X^2] = \int x^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] dx$$

$$= \int \underbrace{(\sigma u + m)^2}_{\sigma^2 u^2 + 2m\sigma u + m^2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$u = \frac{x-m}{\sigma}$

$x = \sigma u + m$

$$= \sigma^2 \underbrace{\int_{\mathbb{R}} u^2 \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du}_{1} + 2m\sigma \underbrace{\int_{\mathbb{R}} u \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du}_{0} + m^2 \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du}_{1}$$

intégration par parties

$$\begin{cases} v' = u\, e^{-u^2/2} du & v = -e^{-u^2/2} \\ w = \frac{u}{\sqrt{2\pi}} & w' = \frac{1}{\sqrt{2\pi}} \end{cases}$$

$$\int_{\mathbb{R}} u^2 p(u) du = \underbrace{\left[-\frac{u}{\sqrt{2\pi}} e^{-u^2/2}\right]_{-\infty}^{+\infty}}_{0} + \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du}_{1}$$

$$\begin{cases} E(X^2) = \sigma^2 + m^2 \\ E(X) = m \end{cases} \Rightarrow \operatorname{Var} X = E(X^2) - E(X)^2$$
$$= \sigma^2 + m^2 - m^2$$
$$= \sigma^2$$

$$\boxed{\operatorname{Var} X = \sigma^2}$$

$$\boxed{X \sim \mathcal{N}(\underset{E(X)}{m}, \underset{\operatorname{Var} X}{\sigma^2})}$$

On centre et on réduit

$$E\left[\frac{X - m}{\sigma}\right] = \frac{1}{\sigma} E[X - m] = \frac{1}{\sigma}(E(X) - m) = \boxed{0}$$

$$\operatorname{Var}\left[\frac{X - m}{\sigma}\right] = \boxed{1}$$
$$= \operatorname{Var}\left[\frac{X}{\sigma} - \frac{m}{\sigma}\right]$$
$$= \operatorname{Var}\left[\frac{X}{\sigma}\right]$$
$$= \frac{1}{\sigma^2} \underbrace{\operatorname{Var} X}_{\sigma^2} = 1$$

Changement de variables

loi de Poisson

$$\boxed{\begin{array}{l} X \sim \mathcal{P}(\lambda) \qquad P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad k \in \mathbb{N} \\ \text{quelle est la loi de } Y = (X-2)^2 \text{ ?} \end{array}}$$

X	Y
0	4
1	1
2	0

- Y prend ses valeurs dans $\{k^2, k \in \mathbb{N}\}$ (carrés des entiers)

3	1
4	4
5	9
6	

• $P[Y = k^2]$?

$$P(Y = k^2] = P[(X-2)^2 = k^2]$$
$$= P[X-2 = k \text{ ou } X-2 = -k]$$
$$= P(X = 2+k \text{ ou } X = 2-k]$$

Cas 1 $2+k = 2-k \iff \boxed{k=0}$

$$P[Y = k^2] = P[Y=0] = P[X=2]$$
$$= \boxed{\frac{\lambda^2}{2} e^{-\lambda}}$$

Cas 2 $\boxed{k \neq 0}$ $P(Y = k^2) = P(X = \underbrace{2+k}_{\in \mathbb{N}}) + P(X = 2-k]$

$$= \frac{\lambda^{2+k}}{(2+k)!} e^{-\lambda} + P(X = 2-k]$$

$2-k \geq 0 \iff \boxed{k \leq 2}$

$$P[X = 2-k] = \begin{cases} 0 & k \geq 3 \\ P[X=0] & k=2 \\ P(X=1] & k=1 \\ P[X=2) & k=0) \text{ déjà traité} \end{cases}$$

Conclusion

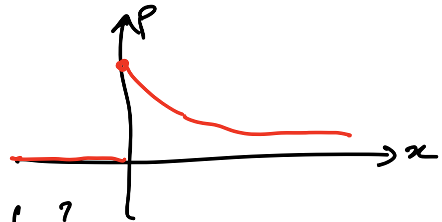
$$\boxed{P(Y = k^2) = \begin{cases} \frac{\lambda^2}{2} e^{-\lambda} & k=0 \\ \frac{\lambda^3}{6} e^{-\lambda} + \lambda e^{-\lambda} & k=1 \\ \frac{\lambda^4}{24} e^{-\lambda} + e^{-\lambda} & k=2 \\ \frac{\lambda^{2+k}}{(2+k)!} e^{-\lambda} & k \geq 3 \ k \in \mathbb{N} \end{cases}}$$

Exemple

$X \sim \mathcal{E}(1)$

Loi exponentielle de paramètre $\lambda = 1$

$p(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases}$



Quelle est la loi de $Y = \frac{1}{X}$?

$$p(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Loi de Y

$$\pi(y) = e^{-\frac{1}{y}} \times \underbrace{\left| \frac{dx}{dy} \right|}_{\text{Jacobien}}$$

$$Y = \frac{1}{X} \iff X = \frac{1}{Y}$$

$$\frac{dx}{dy} = -\frac{1}{y^2}$$

$$\boxed{\pi(y) = \begin{cases} e^{-\frac{1}{y}} \frac{1}{y^2} & y \in \Delta \\ 0 & \text{sinon} \end{cases}}$$

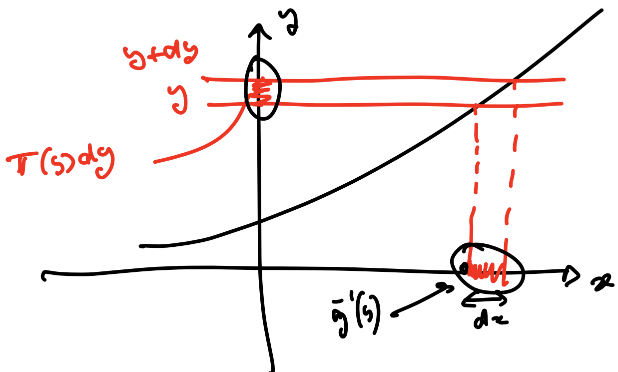
Recherche de Δ : on connait le domaine de X

$$x > 0 \iff \frac{1}{y} > 0 \iff y > 0$$

$$\boxed{\Delta =]0, +\infty[}$$

Idée de preuve

$$y = g(x) \iff x = g^{-1}(y)$$

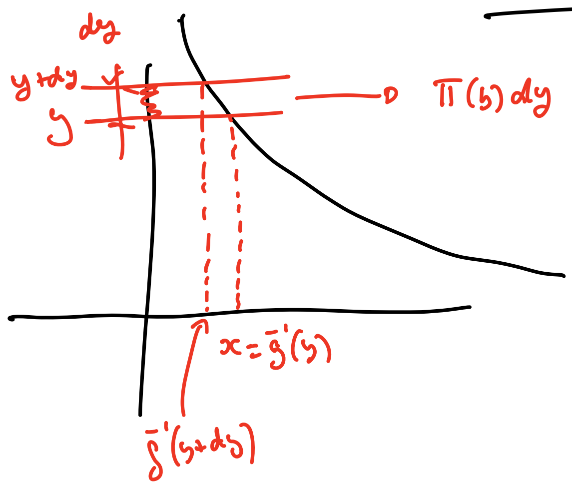


Densité de Y : $P[Y \in]y, y+dy[] \approx \pi(y)\,dy$

//

$$P[Y \in]y, y+dy[] = P[g^{-1}(y)]\, dx$$

donc $\boxed{\pi(y) = p[g^{-1}(y)]\, \underbrace{\frac{dx}{dy}}_{= \left|\frac{dx}{dy}\right|}}$



$= \left|\frac{dx}{dy}\right|$ car $x \to y$ est une fonction croissante

$\boxed{\text{PREUVE}}$

$$P[Y \in \Delta] = \int_\Delta \boxed{\pi(y)}\, dy$$ définition de π

$\forall \Delta$

$$P[Y \in \Delta] = P[g(x) \in \Delta]$$
$$= P[X \in g^{-1}(\Delta)]$$
$$= \int_{g^{-1}(\Delta)} p(x)\, dx$$
$$= \int_\Delta \boxed{p[g^{-1}(y)] \left|\frac{dx}{dy}\right|}\, dy$$

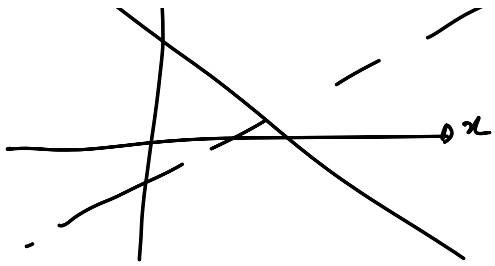
$\in g^{-1}(\Delta)$

$x = g^{-1}(y)$

$y = g(x)$

Exemple 2

$$\boxed{\begin{cases} X \sim N(m, \sigma^2) \\ Y = aX + b \end{cases} \quad \text{Loi de } Y\text{?} \quad a \neq 0}$$



$y = ax + b \iff \boxed{x = \frac{y-b}{a}}$

bijection de $\mathbb{R}$ dans $\mathbb{R}$

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$$

$x \in \mathbb{R}$

Densité de Y

$$\pi(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\left(\frac{y-b}{a} - m\right)^2}{2\sigma^2}\right] \left|\frac{dx}{dy}\right|$$

$$\frac{dx}{dy} = \frac{1}{a}.$$

$$\text{donc} \quad \pi(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{|a|} \exp\left[-\frac{(y-b-am)^2}{2a^2\sigma^2}\right]$$

$$= \frac{1}{\sqrt{2\pi a^2\sigma^2}} \exp\left\{-\frac{[y-(b+am)]^2}{2a^2\sigma^2}\right\}$$

$y = ax + b \Rightarrow E(y) = aE(x) + b = am + b$

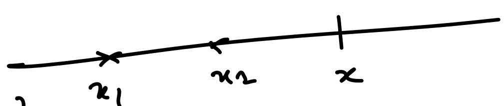
$\mathrm{Var}\, y = a^2 \, \mathrm{Var}\, x = a^2\sigma^2$

$$\boxed{Y \sim N(am+b, a^2\sigma^2)}$$

COURS du 18/09/2023

Exemple d'application où on utilise $F(x) = P[X<x]$

$X = \mathrm{Max}\{X_1, X_2\}$



$P[X<x] = P(X_1<x, X_2<x)$ et

$= P(X_1<x) \; P[X_2<x]$

$$\boxed{F(x) = F_1(x) \; F_2(x)}$$

$$\boxed{\begin{array}{l} X_1 \sim N(m_1, \sigma_1^2) \\ X_2 \sim N(m_2, \sigma_2^2) \\ X_1 \text{ et } X_2 \text{ ind} \end{array} \quad \text{quelle est la loi de } X = X_1 + X_2 \text{ ?}}$$

$$\varphi_X(t) = E[e^{itX}] = E[e^{it(X_1+X_2)}]$$

$$= E[e^{itX_1} \, e^{itX_2}]$$

$$= E[e^{itX_1}] \, E[e^{itX_2}]$$

e^{itX_1} et e^{itX_2} ind

$$= \varphi_{X_1}(t) \, \varphi_{X_2}(t)$$

$$\boxed{\varphi_X(t) = \varphi_{X_1}(t) \, \varphi_{X_2}(t)}$$

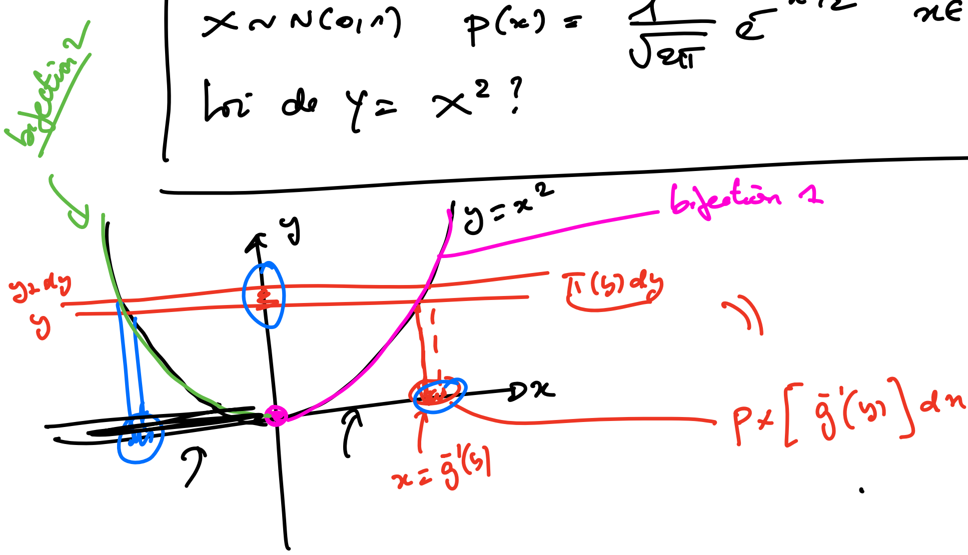
Table

$$e^{im_1 t - \frac{\sigma_1^2}{2} t^2} \qquad e^{im_2 t - \frac{\sigma_2^2}{2} t^2}$$

$$\text{donc} \quad \varphi_X(t) = \exp\left[i(m_1 + m_2) - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) t^2 \right]$$

donc $\boxed{X \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)}$

$$\boxed{\begin{array}{l} X \sim N(0,1) \quad p(x) = \frac{1}{\sqrt{2\pi}} e^{x^2/2} \quad x \in \mathbb{R} \\ \text{Loi de } Y = X^2 \text{ ?} \end{array}}$$



Bijection 1 $\quad \mathbb{R}^{+*} \longleftarrow \mathbb{R}^{+*}$

$x \longrightarrow y = x^2 \iff x = \sqrt{y}$

$\pi_1(y) = \frac{1}{\sqrt{2\pi}} e^{-y/2} \left|\frac{1}{2\sqrt{y}}\right| = \boxed{\frac{1}{2\sqrt{2\pi y}} e^{-y/2}}$

$$\boxed{p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad x \in \mathbb{R}}$$

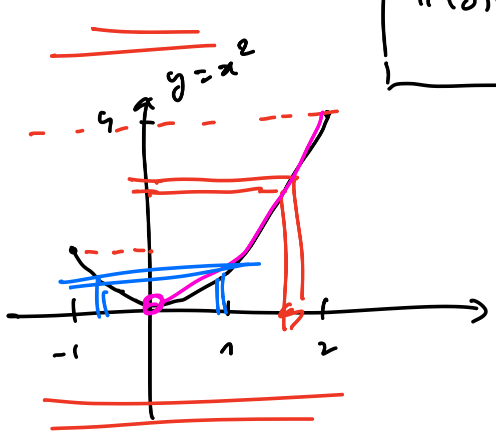
Bijection 2 $\quad \mathbb{R}^{-*} \longrightarrow \mathbb{R}^{+*}$

$x \longrightarrow y = x^2 \iff x = -\sqrt{y}$

$\pi_2(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \left| -\frac{1}{2\sqrt{y}} \right| = \boxed{\frac{1}{2\sqrt{2\pi y}} e^{-y/2}}$

La densité de Y est $\pi(y) = \pi_1(y) + \pi_2(y)$ soit

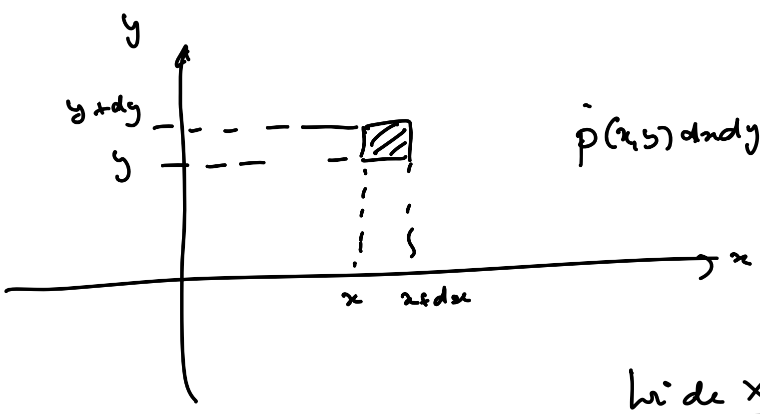
$$\boxed{\pi(y) = \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-y/2} & y \in]0, +\infty[\\ 0 & \text{sinon} \end{cases}}$$



$(X \sim \mathcal{U}(]-1, 2[))$

si $y \in]1, 4[$ alors $\pi(y) = \pi_1(y)$

si $y \in]0, 1[$ alors $\pi(y) = \pi_1(y) + \pi_2(y)$



$p(x,y)\,dx\,dy \simeq P[X \in]x, x+dx[, Y \in]y, y+dy[]$

dx petit
dy petit

Y \ X	0	1
0	1/2	1/6
1	1/6	1/6

Loi de X

$X \to 0$

$X \to 1$

$P[X=0] = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

$P[X=1] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

Δx

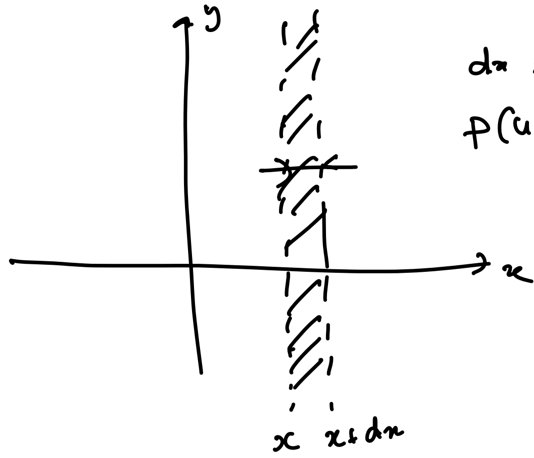
$$\boxed{p(x)dx} \simeq P[X \in]x, x+dx[]$$

$$\|$$

$$P[(X,Y) \in \Delta_x]$$

$$\|$$

$$\int_x^{x+dx} \int_{\mathbb{R}} p(u,v)\, du\, dv$$



dx petit

$p(u,v) \sim p(x,v)$

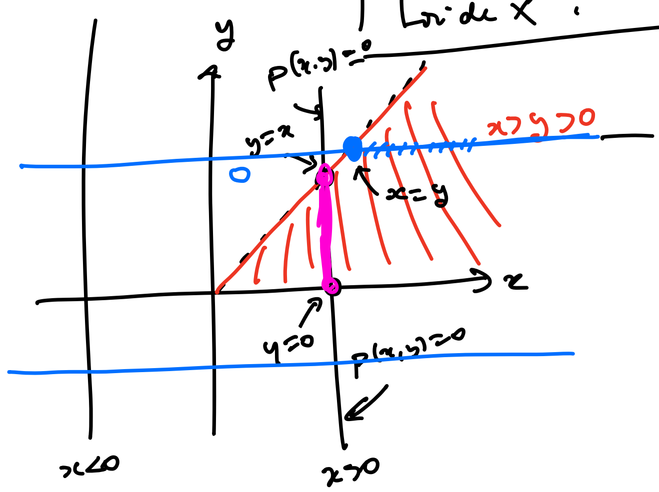
$$\| \quad \int_x^{x+dx} \left[\int_{\mathbb{R}} \underbrace{p(u,v)}_{\simeq p(x,v)}\, dv \right] du = p(x)dx$$

$$\| \quad \int_x^{x+du} \underbrace{\left(\int_{\mathbb{R}} p(x,v)\, dv \right)}_{\text{ind de } u} du = \overbrace{\boxed{\int_{\mathbb{R}} p(x,v)\, dv}}^{p(x)} \underbrace{\int_x^{x+dx} du}_{dx}$$

Exo

$$p(x,y) = \begin{cases} \theta^2 e^{-\theta x} & x > y > 0 \\ 0 & \text{sinon} \end{cases}$$

Loi de X ?



X va va contenir de densité

$$p(x,\cdot) = \int_{\mathbb{R}} p(x,y)\, dy$$

si $x \le 0$ $\quad p(x,\cdot) = \int 0\, dy = \boxed{0}$

si $x > 0$ $\quad p(x,\cdot) = \int_0^x \theta^2 e^{-\theta x}\, dy$

$$= \theta^2 e^{-\theta x} \underbrace{\int_0^x dy}_{x}$$

Conclusion

$$\boxed{\begin{array}{l} p(x,\cdot) = \theta^2 x e^{-\theta x} \, I_{]0,+\infty[}(x) \\ \text{avec} \quad I_{]0,+\infty[}(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} \end{array}}$$

donc $\boxed{X \sim G(\theta, 2)}$

Loi de Y

$$p(\cdot, y) = \int_{\mathbb{R}} p(x,y)\,dx$$

$$= \begin{cases} 0 & y \leq 0 \\ \displaystyle\int_y^{+\infty} \theta^2 e^{-\theta x}\,dx = -\theta\left[e^{-\theta x}\right]_y^{+\infty} = \theta e^{-\theta y} \end{cases}$$

$$\boxed{p(\cdot, y) = \theta e^{-\theta y} \, I_{]0,+\infty[}(y)}$$

TD # 1 du 15/03/2023

$$f(x) = \begin{cases} k x^a & x \in]0,1[\\ 0 & \text{sinon} \end{cases}$$

1) $\displaystyle\int_{\mathbb{R}} f(x)\,dx = 1 \iff k \underbrace{\left[\frac{x^{a+1}}{a+1}\right]_0^1}_{\frac{1}{a+1}} = 1 \iff \boxed{k = a+1}$

Rq : c'est une loi Beta $B(a+1, 1)$

2) $\displaystyle E(X^n) = \int_{\mathbb{R}} x^n f(x)\,dx = \int_0^1 x^n (a+1) x^a\,dx$

$$= (a+1)\left[\frac{x^{a+n+1}}{a+n+1}\right]_0^1 = \boxed{\frac{a+1}{a+n+1}}$$

Moyenne $\boxed{\dfrac{a+1}{a+2} = E(X)}$

Variance $\displaystyle Var(X) = E(X^2) - E^2(X) = \frac{a+1}{a+3} - \left(\frac{a+1}{a+2}\right)^2$

$$= \frac{a+1}{(a+3)(a+2)^2}\left[\underbrace{(a+2)^2-(a+1)(a+3)}_{1}\right]$$

$$= \boxed{\frac{a+1}{(a+3)(a+2)^2}}$$

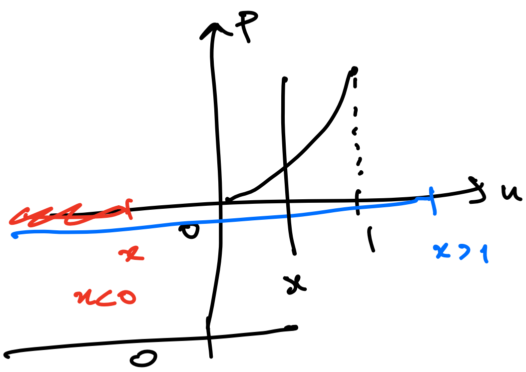
Rq loi Beta (a,b) $\quad \text{Var } X = \frac{ab}{(a+b)^2(a+b+1)}$

$a \to a+1$
$b \to 1$

$$\frac{(a+1)\times 1}{(a+2)^2(a+3)}$$

ok

Fonction de répartition $\quad F(x) = P[X < x] = \int_{-\infty}^{x} p(u)\,du$



$\int_{-\infty}^{x} p(u)\,du = 0 \quad x \leq 0$

$\int_{-\infty}^{x} p(u)\,du = 1 \quad x \geq 1$

Si $x \in]0,1[\quad F(x) = \underbrace{\int_{-\infty}^{0} 0\,du}_{0} + \underbrace{\int_{0}^{x} (a+1)u^a\,du}_{[u^{a+1}]_0^x}$

$= x^{a+1} \qquad x^{a+1} - 0$

$$\boxed{F(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 1 \\ x^{a+1} & x \in]0,1[\end{cases}}$$

changement de variable $\quad Y = -\ln X$

$$f(x) = \begin{cases} (a+1)x^a & x \in]0,1[\\ 0 & \text{sinon} \end{cases}$$

Densité de Y ! $\quad Y = -\ln X$ donc $X = e^{-Y}$)

donc le changement de variable est effectif

Densité de Y $\quad \pi(y) = (a+1)\left(e^{-y}\right)^a \times \left|-e^{-y}\right|$ $\quad \frac{dx}{dy}$

$$= (a+1)\, e^{-ay}\, e^{-y}$$

$$\boxed{\pi(y) = (a+1)\, e^{-(a+1)y} \qquad y \in ??}$$

Domaine de Y

$x \in]0,1[\iff e^{-y} \in]0,1[$

$\iff \ln\left(e^{-y}\right) = -y \in]-\infty, 0[$

$\iff \boxed{y \in]0,+\infty[} \qquad y \in \mathbb{R}^{+*}$

$$\boxed{\pi(y) = (a+1)\, e^{-(a+1)y}\, \mathbb{I}_{]0,+\infty[}(y)}$$

$$\boxed{Y \sim \mathrm{Ga}(a+1, 1)}$$

$E(Y) = \frac{\nu}{\theta} = \boxed{\frac{1}{a+1}}$

$Var(Y) = \frac{\nu}{\theta^2} = \boxed{\frac{1}{(a+1)^2}}$

$$E[Y^n] = \int_0^{+\infty} y^n (a+1)\, e^{-\overset{u}{\boxed{(a+1)y}}}\, dy$$

$u = (a+1)y$

$y = \frac{u}{a+1}$

$$= \int_0^{+\infty} \frac{u^n}{(a+1)^n} \cancel{(a+1)}\, e^{-u} \frac{du}{\cancel{(a+1)}}$$

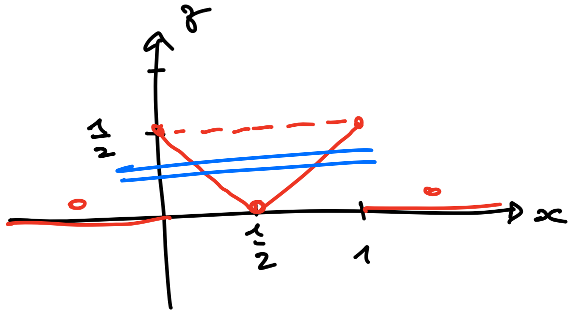
$$= \frac{1}{(a+1)^n} \underbrace{\int_0^{+\infty} u^n e^{-u}\, du}_{\Gamma(n+1) = n!} = \boxed{\frac{n!}{(a+1)^n}}$$

6) $Z = \left|X - \frac{1}{2}\right|$

Loi de Z ?

$$f(x) = \begin{cases} (a+1)x^a & x \in]0,1[\\ 0 & \text{sinon} \end{cases}$$

$$\left|x - \frac{1}{2}\right| = \begin{cases} x - \frac{1}{2} & x \geq \frac{1}{2} \\ \frac{1}{2} - x & x \leq \frac{1}{2} \end{cases}$$



donc $Z \in]0, \frac{1}{2}[$

Bijection # 1

$$]0, \tfrac{1}{2}[\longrightarrow]0, \tfrac{1}{2}[$$

$$x \longmapsto z = \left|x - \tfrac{1}{2}\right| = \tfrac{1}{2} - x \qquad \left|\frac{dx}{dz}\right| = |-1| = 1$$

$$x = \tfrac{1}{2} - z$$

$$g_1(z) = (a+1)\left(\tfrac{1}{2} - z\right)^a \times 1 = (a+1)\left(\tfrac{1}{2} - z\right)^a$$

Bijection # 2

$$]\tfrac{1}{2}, 1[\longrightarrow]0, \tfrac{1}{2}[$$

$$x \longmapsto z = \left|x - \tfrac{1}{2}\right| = x - \tfrac{1}{2} \qquad \left|\frac{dx}{dz}\right| = |1| = 1$$

$$x = \tfrac{1}{2} + z$$

$$g_2(z) = (a+1)\left(\tfrac{1}{2} + z\right)^a \times 1$$

$$= (a+1)\left(z + \tfrac{1}{2}\right)^a$$

Donc la densité de Z est

$$\boxed{g(z) = \begin{cases} (a+1)\left[\left(\frac{1}{2} - z\right)^a + \left(\frac{1}{2} + z\right)^a\right] & z \in]0, \frac{1}{2}[\\ 0 & \text{sinon} \end{cases}}$$

Files d'attente

A — Guichet 1

B — Guichet 2

C

$P[T<t]$

Temps d'attente de C = Inf (temps de service de A , temps de service de B)

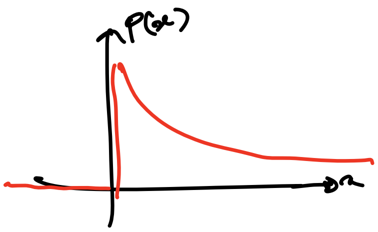
$T = \mathrm{Inf}(X, Y)$

$E(T)$?

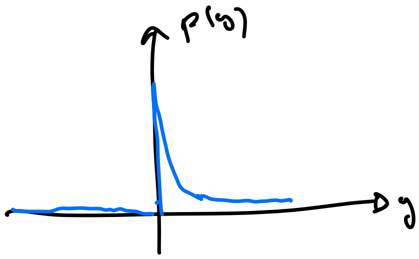
$E[e^{iuT}]$

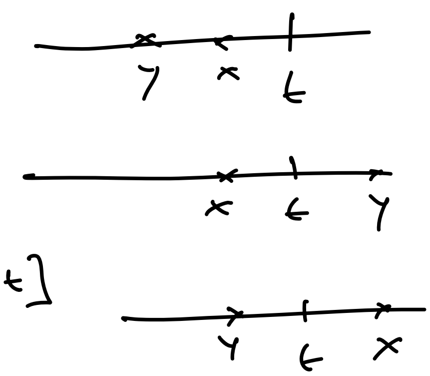
$E[e^{iu\,\mathrm{Inf}(X,Y)}]$

$X \sim \mathcal{E}(\lambda)$ $\quad p(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$



$Y \sim \mathcal{E}(\mu)$ $\quad p(y) = \begin{cases} \mu e^{-\mu y} & y > 0 \\ 0 & y \leq 0 \end{cases}$





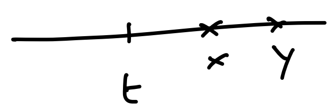
Méthode 1

$P[T<t] = P[\mathrm{Inf}(X,Y) < t]$

$= P[\, X<t, Y<t$ ou $X<t, Y>t$ ou $Y<t, X>t \,]$

$= \ldots$

Méthode 2



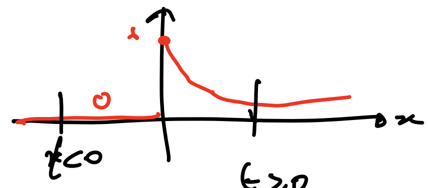
$P[T<t] = 1 - P[T \geq t]$

$= 1 - P[X \geq t, Y \geq t]$

$$= 1 - \underbrace{P[X \geq t]}_{1 - F_X(t)} \; \underbrace{P[Y \geq t]}_{1 - F_Y(t)}$$

$X \perp Y$ ind

$$F_X(t) = P[X < t] \qquad X \sim \mathcal{E}(\lambda) \qquad p(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$$



$$= \begin{cases} 0 & t < 0 \\ \int_0^t \lambda e^{-\lambda x} \, dx = \left[- e^{-\lambda x} \right]_0^t = 1 - e^{-\lambda t} \end{cases}$$

$$P[T < t] = 1 - \underbrace{e^{-\lambda t}}_{1 - F_X(t)} \; \underbrace{e^{-\mu t}}_{1 - F_Y(t)} = 1 - e^{-(\lambda + \mu) t}$$

densité de T
$$\boxed{p(t) = \begin{cases} (\lambda + \mu) e^{-(\lambda + \mu) t} & t > 0 \\ 0 & < 0 \end{cases}}$$

Tard $\boxed{E(T) = \frac{1}{\lambda + \mu}}$

Cours du 25/09/2023

indépendance $\quad p_{ij} = p_{i\cdot} \times p_{\cdot j} \quad \forall i \forall j$ //

$$P[X \in \underbrace{\{x_1, x_2\}}_{\Delta}, Y \in \underbrace{\{y_1\}}_{\Delta'}] = P[(X,Y) = (x_1, y_1) \text{ ou } (X,Y) = (x_2, y_1)]$$

$$= p_{11} \times p_{21}$$

$$= p_{1\cdot} \, p_{\cdot 1} \times p_{2\cdot} \, p_{\cdot 1}$$

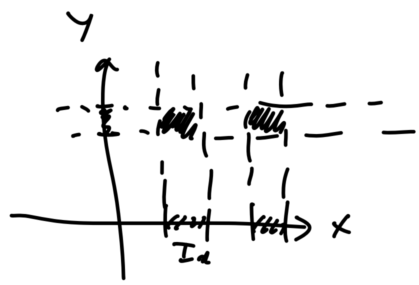
$$= \underbrace{(p_{1\cdot} + p_{2\cdot})}_{} \; \underbrace{p_{\cdot 1}}_{P[Y = y_1]}$$

$p[X \in (x_1, x_2)]$

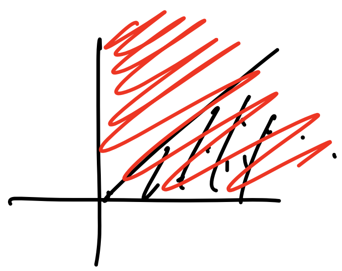
Cas continu

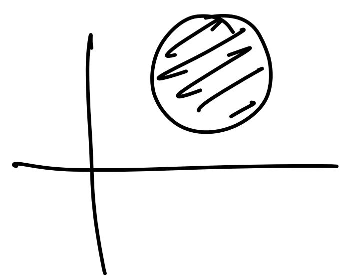
$$X \text{ et } Y \text{ ind} \iff p(x,y) = \underbrace{p(x,\cdot)}_{x \in I_x} \times \underbrace{p(\cdot,y)}_{y \in I_y} \quad \forall x \forall y$$

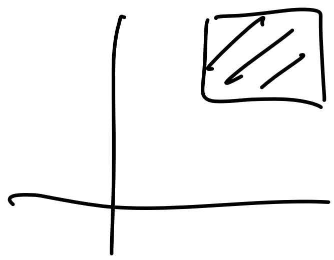
Eg

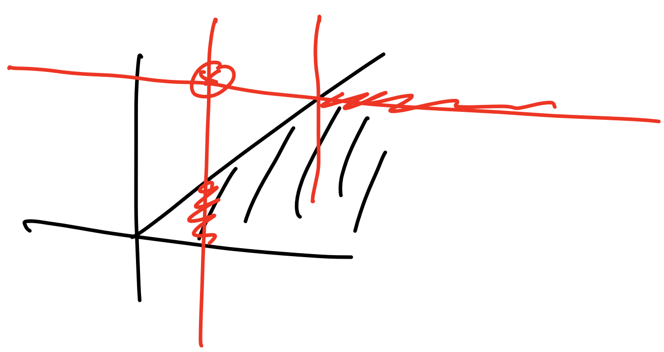


$\boxed{\text{le domaine de définition de } (X,Y) \text{ est une réunion de pavés}}$









Espérances mathématiques d'un couple (X,Y)

à 1D $\quad E[X^2] \to \sum_i x_i^2 \, p[X = x_i]$

$\quad\quad\quad\quad\quad\;\; \to \int x^2 \underbrace{p(x,\cdot)} dx$

à 2D

$$E[XY] \to \sum_{i,j} x_i y_j \underbrace{P[X=x_i, Y=y_j]}_{p_{ij}}$$

$$E[XY] \to \iint xy \underbrace{p(x,y)}\, dx\,dy$$

$$E[\alpha(X,Y)] \to \sum_{i,j} \alpha(x_i, y_j)\, p_{ij}$$

$\forall \alpha$

$$E[\alpha(X,Y)] \to \iint \alpha(x,y)\, p(x,y)\, dx\,dy$$

Propriétés

$$E[\text{cste}] = \sum_{i,j} \text{cte}\ p_{ij} = \text{cte}$$

$$E[\text{cste}] = \iint \text{cte}\ p(x,y)\,dx\,dy = \text{cte}$$

$$E\left[XY - \frac{X^2}{Y^2}\right] = E[XY] - E\left[\frac{X^2}{Y^2}\right]$$

E est un opérateur linéaire

Définition cohérente (X,Y) couple de va continues.

$$E[\underbrace{X}_{\alpha(X,Y)}] = \iint x\, p(x,y)\,dx\,dy = \int_x \Big[\underbrace{\int_y x\, p(x,y)\,dy}\Big]dx$$

$$x \int p(x,y)\,dy$$

$$\left(E[\underbrace{X}_{\alpha(X)}] = \int x\ p(x,\cdot)\,dx \right.$$

Indépendance

$$\boxed{\text{si } X \text{ et } Y \text{ sont ind alors } E[\alpha(X)\beta(Y)] = E[\alpha(X)]\,E[\beta(Y)] \quad \forall \alpha,\beta}$$

En effet (cas continu)

$$E[\alpha(X)\beta(Y)] = \iint \alpha(x)\beta(y)\ p(x,y)\,dx\,dy$$

$$= \iint \alpha(x)\,\beta(y)\ p(x,\cdot)\,p(\cdot,y)\,dx\,dy$$

X et Y ind

$$= \left[\int \alpha(x)\,p(x,\cdot)\,dx\right] \times \left[\int \beta(y)\,p(\cdot,y)\,dy\right]$$

$= E(\alpha(X))\ E[\beta(Y)]$

Exemple d'application

$$\boxed{\text{si } X \text{ et } Y \text{ sont indépendantes, alors } E[XY] = E(X)E(Y)}$$

Définition

$$\boxed{\begin{array}{c} Cov(X,Y) = E(XY) - E(X)E(Y) \\ \text{s'appelle la covariance du couple } (X,Y) \end{array}}$$

Rq si $X=Y$ $\quad Cov(X,X) = \underline{Var X}$

Matrice de covariance

$$M = \begin{pmatrix} Var X & Cov(X,Y) \\ Cov(X,Y) & Var Y \end{pmatrix}$$

$$V = \begin{bmatrix} X - E(X) \\ Y - E(Y) \end{bmatrix} \quad \text{vecteur de } \mathbb{R}^2$$

$$E[VV^T] = E\Big[\underbrace{\begin{pmatrix} X - E(X) \\ Y - E(Y) \end{pmatrix} (X - E(X), Y - E(Y))}\Big]$$

$$= E\left[\begin{pmatrix} (X-E(X))^2 & (X-E(X))(Y-E(Y)) \\ (Y-E(Y))(X-E(X)) & (Y-E(Y))^2 \end{pmatrix}\right]$$

$$= \begin{pmatrix} Var X & cov(X,Y) \\ cov(X,Y) & Var Y \end{pmatrix}$$

à 1D $\quad$ Fonction caractéristique $\quad \boxed{E[e^{iuX}]} = \varphi_X(u)$

à 2D $\quad E\left[e^{i(u_1 X + u_2 Y)}\right] = \varphi_{X,Y}(u) \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$$i(u_1 X + u_2 Y) \longrightarrow (u_1, u_2)\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\boxed{\varphi_{X,Y}(u) = E\left[e^{iu^T\binom{X}{Y}}\right]}$$

Coefficient de corrélation

$$r_{X,Y} = \frac{cov(X,Y)}{\sigma_X \sigma_Y} \qquad \sigma_X = \sqrt{Var X} \qquad \sigma_Y = \sqrt{Var Y}$$

$$r_{10X,10Y} = \frac{cov(10X,10Y)}{\sqrt{Var(10X)}\sqrt{Var(10Y)}} = \frac{100\; cov(X,Y)}{\sqrt{100 Var X}\sqrt{100 Var Y}} = r_{X,Y}$$

Rappel : Inégalité de CAUCHY-SCHWARZ

$$|\langle A, B\rangle| \le \sqrt{\|A\|^2}\sqrt{\|B\|^2} = \|A\|\;\|B\|$$

$\sum a_i b_i$

$\|a\|^2 = \sum a_i^2$

On démontre que si X et Y sont des variables aléatoires $\langle X, Y\rangle = E[XY]$ définit un produit scalaire

$$\|X\|^2 = \langle X, X\rangle = E(X^2)$$

Cauchy-Schwartz

$$|\underbrace{\langle X - E(X), Y - E(Y)\rangle}_{cov(X,Y)}| \le \|X - E(X)\| \times \|Y - E(Y)\|$$

$$\le \sqrt{E[(X-E(X))^2]} \times \sqrt{E[(Y-E(Y))^2]}$$

$$\boxed{r_{X,Y} = \frac{cov(X,Y)}{\sigma_X \sigma_Y}}$$

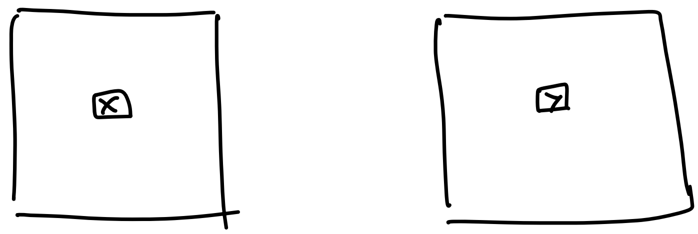
$$r_{XY}^2 = \frac{cov^2(X,Y)}{\sigma_X^2 \sigma_Y^2}$$

$$|cov(X,Y)| \le \sqrt{Var X}\sqrt{Var Y}$$

d'où
$$\frac{|cov(X,Y)|}{\sqrt{\sigma_X^2}\sqrt{\sigma_Y^2}} \le 1$$

$$\boxed{r_{XY}^2 \le 1} \iff \boxed{-1 \le r_{XY} \le +1}$$

$r_{XY} = \pm 1 \iff X$ et Y liés par une relation affine



I_1 I_2

$$r_{xy} = \frac{cov(x,y)}{\sigma_x \sigma_y} = \frac{E[xy] - E[x]E[y]}{\sqrt{E[x^2] - E^2[x]}\,\sqrt{E[y^2] - E^2[y]}}$$

les 2 pixels x et y sont liés s'il n'y a pas de changement

les 2 ―――――― peu liés s'il y a eu un changement

d'où la règle

$\boxed{\text{Détecter un changement si } r^2_{x,y} \le \text{seuil}}$

fixé par l'utilisateur

$E(x)$?

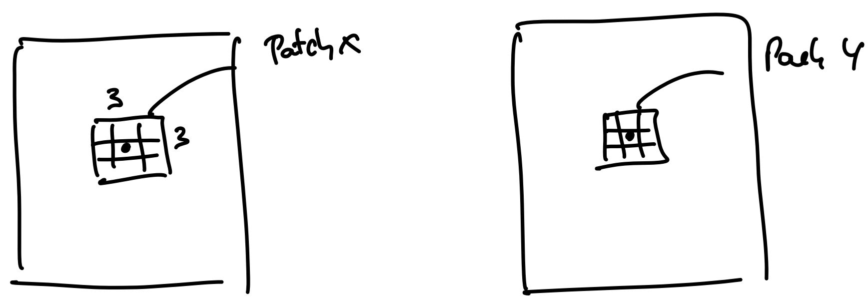
$E(y)$?

$E(x^2)$?

$E(y^2)$?

$E(xy)$?

$$\Rightarrow r_{x,y} = \frac{E(xy) - E(x)E(y)}{\sqrt{E(x^2) - E^2[x]}\,\sqrt{E[y^2] - E^2[y]}}$$



$I_1 = x$ $I_2 = y$

$E(x) \simeq \frac{1}{9} \sum_{x_{ij} \in \text{Patch}} x_{ij}$,

$E(y) \simeq \frac{1}{9} \sum y_{ij}$

$E(x^2) \simeq \frac{1}{9} \sum x^2_{ij}$

$E(y^2) \simeq \frac{1}{9} \sum y^2_{ij}$

$E(xy) \simeq \underbrace{\frac{1}{9} \sum x_{ij} y_{ij}}$

$$\mathrm{Cov}(X,Y) = \underbrace{\frac{1}{n}\sum x_{ij}y_{ij}}_{\overline{x_{ij}y_{ij}}} - \Big(\underbrace{\frac{1}{n}\sum x_{ij}}_{\overline{x_{ij}}}\Big)\Big(\underbrace{\frac{1}{n}\sum y_{ij}}_{\overline{y_{ij}}}\Big) = 0$$

Espérance conditionnelle

$$p(y|x) = \frac{p(y,x)}{p(x,\cdot)}$$

$$E[\alpha(X,Y)] = \iint \alpha(x,y)\underbrace{p(x,y)}_{p(y|x)\,p(x,\cdot)}\,dx\,dy$$

$$= \int_x \left[\int_y \alpha(x,y)\,p(y|x)\,p(x,\cdot)\,dy\right] dx$$

$$= \int_x p(x,\cdot)\underbrace{\left[\int_y \alpha(x,y)\,p(y|x)\,dy\right]}_{E_Y[\alpha(X,Y)\,|\,X=x]} dx$$

$$E(\alpha(X)) = \int \alpha(x)\,p(x,\cdot)\,dx$$

$$\boxed{= E_X\left[E_Y(\alpha(X,Y)\,|\,X)\right]}$$

Exemple d'application

$X \sim N(0,1)$

$p(x,\cdot) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ $x \in \mathbb{R}$

$Y \to 1$ p

$Y \to -1$ $q = 1-p$

Quelle est la loi de $Z = XY$?

Fonction caractéristique de Z

$$\varphi_Z(u) = E[e^{iuZ}] = E[e^{iuXY}]$$

$$= E_Y\left(E_X(e^{iuXY}\,|\,Y)\right)$$

TABLES

$$E\left[e^{iuXY} \mid Y\right] = \varphi_X(uY) = e^{-\frac{1}{2}u^2Y^2}$$

$X \sim N(0,1)$

$$\varphi_X(t) = e^{imt - \frac{\sigma^2 t^2}{2}} = e^{-t^2/2}$$

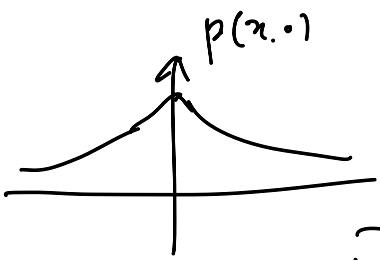
$m=0$

$\sigma^2=1$

$$\varphi_Z(u) = E_Y\left[e^{-\frac{u^2Y^2}{2}}\right]$$

$$= e^{-\frac{u^2}{2}(1)^2} P(Y=1) + e^{-\frac{u^2}{2}(-1)^2} P(Y=-1)$$

$$\boxed{\varphi_Z(u) = e^{-u^2/2}} \Rightarrow \boxed{Z \sim N(0,1)}$$



Exemple 2

$X_i \to 1 \quad p$

$X_i \to 0 \quad q = 1-p$

$$Y_N = \sum_{i=1}^{N} X_i$$

$N \sim P(\lambda) \qquad P(N=k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad k \in \mathbb{N}$

$E(Y_N) = ?$

Si $N = n$ alors on sait que $Y_n = \sum_{i=1}^{n} X_i \sim B(n,p)$

<u>Table</u> $\boxed{E(Y_n) = np}$

$$E[Y_N] = E_N\Big[\underbrace{E[Y_N|N]}_{Np}\Big]$$

$$= p\,\underbrace{E_N[N]}_{\text{Espérance d'une loi de Poisson } = \lambda}$$

$$= \boxed{\lambda p}$$

<u>TD #2 du 25/09/2023</u>

<u>Rappel</u> | Si $X \sim N(m, \sigma^2)$ alors

$$Y = aX + b \sim N(\underbrace{am+b}_{E(Y)}, \underbrace{a^2\sigma^2}_{\text{Var}\,Y})$$

$$\boxed{Z = \frac{X - m}{\sigma} \sim N(0,1)}$$

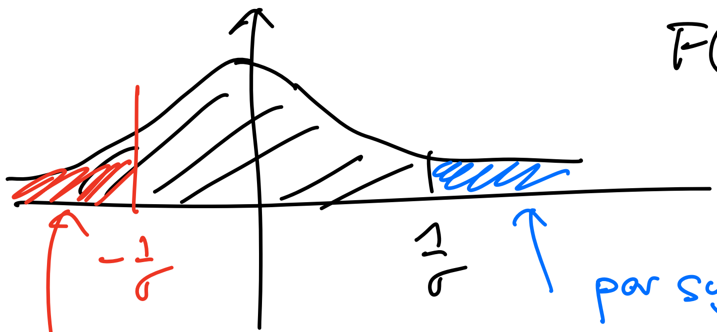
on centre et on réduit !!

Fonction de répartition de Z

$$P[Z < z] = \int^{z} \frac{1}{\sqrt{\ }}\, e^{-u^2/2}\, du$$

$= F(z)$

normcdf.m ← cumulative distribution function

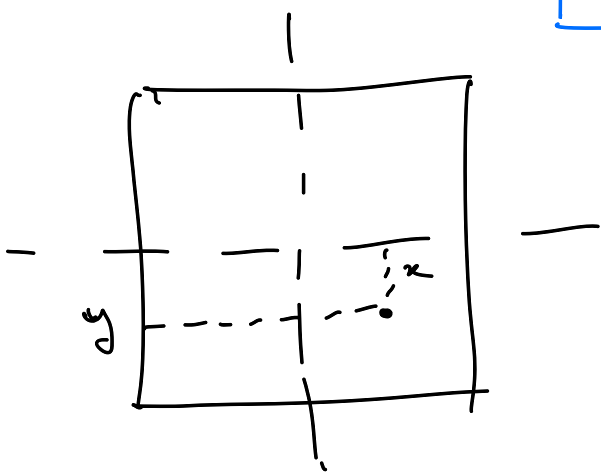


$F\left(\frac{1}{\sigma}\right)$

par symétrie $= 1 - F\left(\frac{1}{\sigma}\right)$

donc $\boxed{F\left(-\frac{1}{\sigma}\right) = 1 - F\left(\frac{1}{\sigma}\right)}$

$F\left(-\frac{1}{\sigma}\right)$



i) $q = P[-1 \leq X \leq 1,\ -1 \leq Y \leq 1]$

$= P[-1 \leq X \leq 1] \times P[-1 \leq Y \leq 1]$

X et Y ind

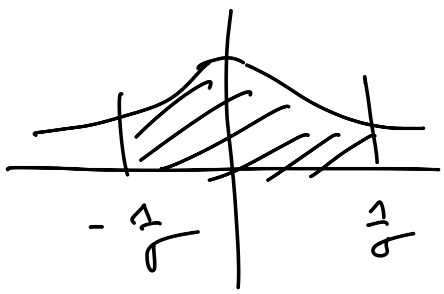
$= P\left[-\frac{1}{\sigma} \leq Z \leq \frac{1}{\sigma}\right] P\left[-\frac{1}{\sigma} < \frac{Y}{\sigma} < \frac{1}{\sigma}\right]$

$\boxed{\text{Méthode 1}}$

$$Z = \frac{X-0}{\sigma} \sim N(0,1)$$

donc $P[Z<z] = F(z)$

$$= \underbrace{\left(\int_{-\frac{1}{\sigma}}^{\frac{1}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du\right)}_{F(\frac{1}{\sigma})-F(-\frac{1}{\sigma})} \times \left(F(\tfrac{1}{\sigma}) - F(-\tfrac{1}{\sigma})\right)$$



$$\boxed{q = \left[F(\tfrac{1}{\sigma}) - F(-\tfrac{1}{\sigma})\right]^2}$$

pour $\sigma = \frac{1}{2}$ $\quad \boxed{q = \left(F(2) - F(-2)\right)^2}$

$\boxed{\text{Méthode 2}}$

$$P[-1 \le X \le 1] = \int^{+1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}}\,du$$

$$= \int_{-\frac{1}{\sigma}}^{\frac{1}{\sigma}} \frac{1}{\sqrt{2\pi\cancel{\sigma^2}}} e^{-\frac{v^2}{2}} \cancel{\sigma}\,dv$$

$v = \frac{u}{\sigma}$

$$= \boxed{F(\tfrac{1}{\sigma}) - F(-\tfrac{1}{\sigma})}$$

$p =$ Probabilité que la flèche n'atteigne pas le carton.

$=$ 1 − Probabilité que la flèche atteigne le carton

$$\boxed{p = 1 - q}$$

2) $N \in \{0, \dots, n\}$ = nombre de flèche perdues
n tirs

sur 1 tir $\quad P[\text{flèche perdue}] = p$

$P[\text{flèche non perdue}] = 1 - p$

si X_k sur le résultat au $k^{\text{ème}}$ tir

succès.

$$X_k \begin{cases} \to 1 & p \\ \to 0 & q = 1 - p \end{cases}$$

$$N = \sum_{k=1}^{n} X_k$$

Rappel. $P\left[\begin{matrix} k \text{ succès} \\ \text{sur } n \text{ expériences} \end{matrix}\right] = P[Z = k]$

$Z \sim B(n, p_s)$

nombre d'expérience

p_s probabilité du succès sur 1 expérience

$$\boxed{\begin{array}{l} N \sim B(100, p) \qquad p = 1-q \\ E[N] \underset{\underset{\text{Table}}{\uparrow}}{=} 100p \end{array}}$$

Exo 1

$$Y \mid X = n \sim B(n, p)$$

donc $P[Y=k \mid X=n] = \binom{n}{k} p^k q^{n-k}$

$\boxed{k \in \{0, \dots, n\}}$

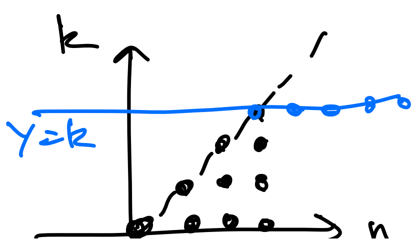
$X \sim P(\lambda)$ donc $P[X=k] = \dfrac{\lambda^k}{k!} e^{-\lambda}$

$k \in \mathbb{N}$

---

1) Loi de (X, Y)

$$P[Y=k,\ X=n] = P(Y=k \mid X=n)\, P(X=n)$$



$$= \boxed{\binom{n}{k} p^k q^{n-k} \frac{\lambda^n}{n!} e^{-\lambda}}$$

probabilité

$$\boxed{k \in \mathbb{N} \qquad k \le n}$$

Domaine

associées

2) Loi de Y

$$P[Y=k] = \sum_n P[Y=k,\ X=n]$$

$$= \sum_{n=k}^{+\infty} \frac{\cancel{n!}}{k!(n-k)!}\, p^k \frac{q^{n-k} \lambda^n e^{-\lambda}}{\cancel{n!}}$$

$$= \frac{1}{k!} p^k e^{-\lambda} \sum_{n=k}^{+\infty} \frac{q^{n-k} \lambda^n}{(n-k)!}$$

$$= \frac{p^k}{k!} e^{-\lambda} \sum_{m=0}^{+\infty} \frac{q^m \lambda^{m+k}}{m!}$$

$m = n-k$

$n = m+k$

$$= \frac{\lambda^k p^k e^{-\lambda}}{k!} \underbrace{\sum_{m=0}^{+\infty} \frac{(\lambda q)^m}{m!}}_{e^{\lambda q}}$$

$$= \frac{(\lambda p)^k}{k!} e^{-\lambda(\underbrace{1-q}_{p})}$$

$$\boxed{Y \sim P(\lambda p)}$$

Si $p = \frac{1}{2}$ $\qquad \boxed{Y \sim \mathcal{P}\left(\frac{\lambda}{2}\right)}$

3) Calcul de $E[XY]$

Méthode 1 $\qquad E[XY] = \sum_n \sum_k nk\, P[X=n, Y=k]$

$$\binom{n}{k} p^k q^{n-k} \frac{\lambda^n}{n!} e^{-\lambda}$$

$$k \in \mathbb{N} \qquad n \geq k$$

$$= \sum_{k=0}^{+\infty} \sum_{n=k}^{+\infty} \ldots$$

pénible

Méthode 2 $\qquad E(XY) = E_X\left[E_Y(XY \mid X=n) \right]$

$$= E_X\Big[\underbrace{n}_{\cancel{X}}\, E[Y \mid X=n] \Big]$$

$$Y \mid X=n \sim B(n,p) \xRightarrow[\text{Table}]{} E[Y \mid X=n] = np$$

$$= E_X[\, n \times np \mid X = n]$$

$$= p\, E[X^2]$$

$X \sim P(\lambda)$ donc $E(X^2) = \mathrm{Var} X + E^2[X]$

car $\mathrm{Var} X = E[X^2] - E^2[X]$

λ $\quad$ λ^2

$p = \frac{1}{2}$ donc $E(XY) = \frac{1}{2} \times (\lambda + \lambda^2)$

$$= \boxed{\frac{\lambda(1+\lambda)}{2}}$$

4) a?

$$E[\alpha(X,Y)] = E[E(\alpha(X,Y) \mid X)]$$

$$\mathrm{cov}(X,Z) = E(XZ) - E(X)\,E(Z)$$

$Z = X + aY$

E $\quad X \sim P(\lambda) \Rightarrow E[X] = \lambda$

$Z = X + aY \Rightarrow E[Z] = \underbrace{E[X]}_{\lambda} + a E[Y]$

$\frac{\lambda}{2}$

car $Y \sim P(\lambda/2)$

donc $\begin{cases} E[X] = \lambda \\ E[Z] = \lambda + \frac{a\lambda}{2} \end{cases}$

$$E[XZ] = E[X^2 + aXY]$$
$$= \underbrace{\mathrm{Var}\,X}_{\lambda} + \underbrace{E^2[X]}_{\lambda^2} + a\,E(XY)$$

vu à la question précédente

$\frac{\lambda(\lambda+1)}{2}$

On cherche a /

$$\underbrace{E[XZ]} - E(X)E[Z] = 0$$

$$\lambda + \cancel{\lambda^2} + a\,\frac{\lambda(\lambda+1)}{2} - \lambda\left(\cancel{\lambda} + \frac{a}{2}\lambda\right) = 0$$

$$1 + \frac{a}{2}(\cancel{\lambda}+1) - \frac{a}{2}\cancel{\lambda} = 0$$

$$\frac{a}{2} + 1 = 0 \qquad \boxed{a = -2}$$

Cours du 21/10/2023

ex 1D

$$p(x) = \begin{cases} 2x & x \in]0,1[\\ 0 & \text{sinon} \end{cases}$$

loi de $Y = \ln X$? $\quad X = e^Y$

loi de Y

densité de Y

$$\pi(y) = \begin{cases} 2e^y \, |e^y| & \\ 0 \text{ sinon} \end{cases} \qquad y \in ? \quad \frac{dx}{dy}$$

$$\boxed{\pi(y) = 2e^{2y} \quad y \in ?}$$

$x \in]0,1[\Rightarrow y = \ln x \in \mathbb{R}^{-*} =]-\infty, 0[$

$$\boxed{\pi(y) = 2e^{2y} \, \mathbb{1}_{]-\infty,0[}(y)}$$

Preuve.

$$P\left(\binom{U}{V} \in \Delta\right) = \iint_\Delta \pi(u,v) \, du \, dv$$

$$P\left[\begin{pmatrix} U \\ V \end{pmatrix} \in \Delta\right] = P\left[g\begin{pmatrix} X \\ Y \end{pmatrix} \in \Delta\right]$$

$$= P\left[\begin{pmatrix} X \\ Y \end{pmatrix} \in g^{-1}(\Delta)\right]$$

VD !!

$$= \iint_{g^{-1}(\Delta)} p(x,y)\, dx\, dy$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = g\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{———} \quad = \iint_{\Delta} p\left[g^{-1}(u,v)\right] \left|\det(J)\right| du\, dv$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

donc $\boxed{\pi(u,v) = p\left[g^{-1}(u,v)\right] \left|\det(J)\right|}$

Exemple 1

$X \sim U(]0,1[)$

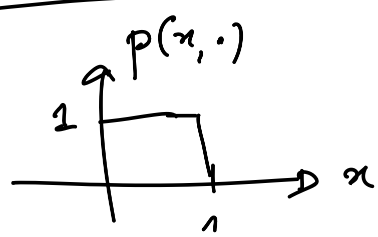
$Y \sim U(]0,1[)$

X et Y indépendants

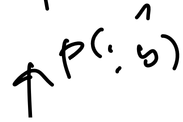
Loi de $\begin{pmatrix} T \\ U \end{pmatrix}$?

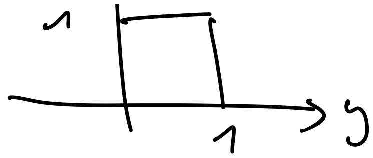
$$\begin{cases} T = X + Y \\ U = X \end{cases}$$

$X \sim U(]0,1[)$

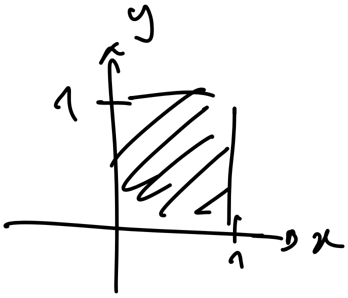


$Y \sim U(]0,1[)$





X et Y ind $\Rightarrow$ $p(x,y) = p(x,\cdot) \times p(\cdot,y)$



$$= \begin{cases} 1 & \text{si } (x,y) \in]0,1[\times]0,1[\\ 0 & \text{sinon} \end{cases}$$

<u>Densité de</u> $\begin{pmatrix} T \\ U \end{pmatrix} = \begin{pmatrix} X+Y \\ X \end{pmatrix} \Rightarrow \begin{cases} X = U \\ Y = T - U \end{cases}$

donc le changement de variable est bijectif.

$$\pi(t,u) = \underbrace{1}_{\substack{\text{1 dans lequel j'ai remplacé} \\ (x,y) \text{ par } (u,\, t-u)}} \times |\det(J)|$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$\det(J) = -1 \Rightarrow |\det(J)| = 1$

donc $\boxed{\pi(t,u) = 1 \times 1 \quad (t,u) \in \Delta}$

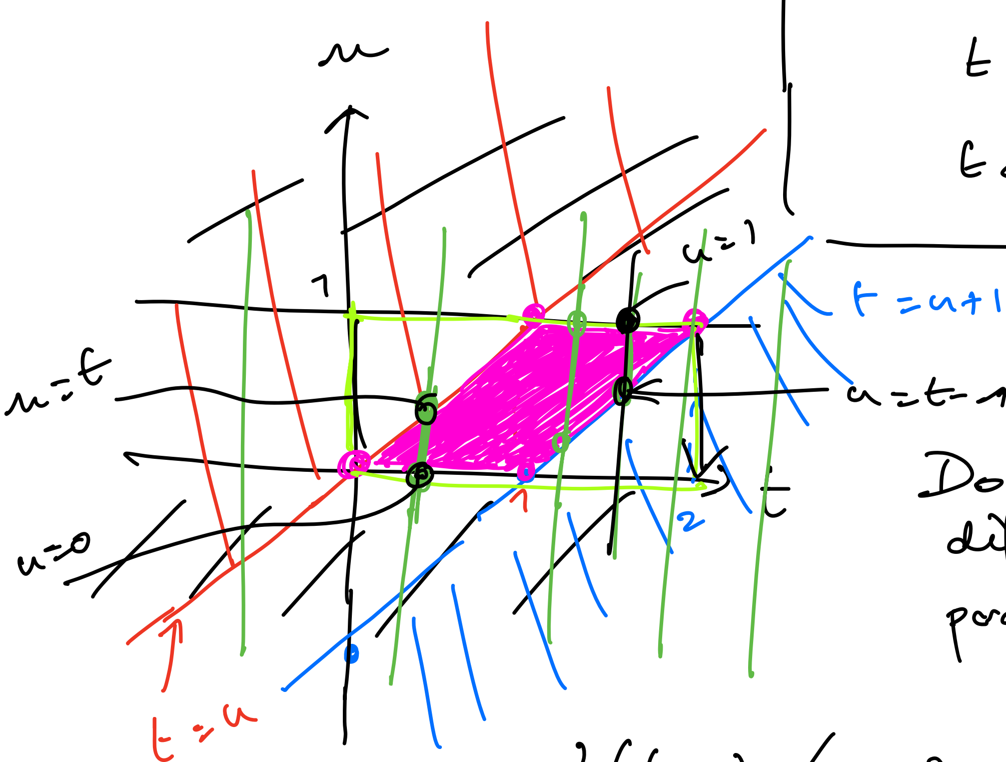
Recherche de Δ

$$\begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases} \iff \begin{cases} 0 < u < 1 \\ \underbrace{0 < t-u}_{} < 1 \end{cases}$$

Domaine de (X,Y)

$\iff$

$$\boxed{\begin{array}{c} u > 0 \\ u < 1 \\ t > u \\ t < u+1 \end{array}}$$



Domaine de définition = parallélogramme

$\{(t,u) \,/\, u \in]0,1[,\ t > u,\ t < u+1\}$

Lois marginales de T et de U

$$\begin{cases} T = X+Y \\ U = X \end{cases}$$

U a la même loi que X donc $\boxed{U \sim \mathcal{U}(]0,1[)}$

Loi de T

D'après la figure, T est à valeurs dans $]0,2[$

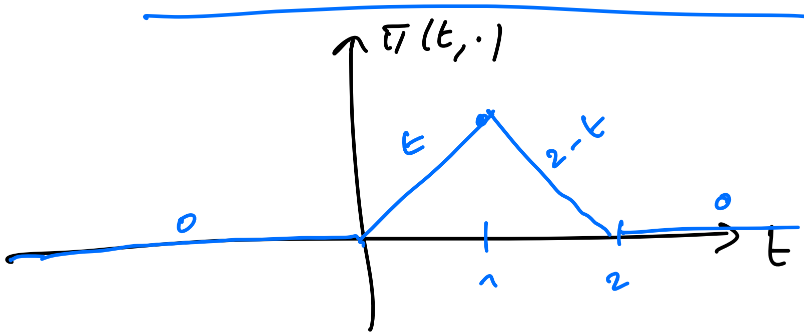
La densité de T est

$$\pi(t,\cdot) = \int \pi(t,u)\,du$$

$$= \begin{cases} 0 & \text{si } t \notin]0,2[\\ \int_0^t \underbrace{\pi(t,u)}_{1}\,du & t \in]0,1] \\ \int_{t-1}^1 \underbrace{\pi(t,u)}_{1}\,du & t \in]1,2[\end{cases}$$

donc

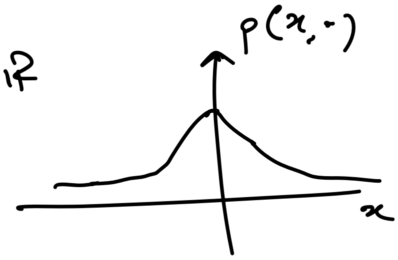
$$\boxed{\pi(t,\cdot) = \begin{cases} 0 & t \notin]0,2[\\ t & t \in]0,1] \\ 2-t & t \in]1,2[\end{cases}}$$



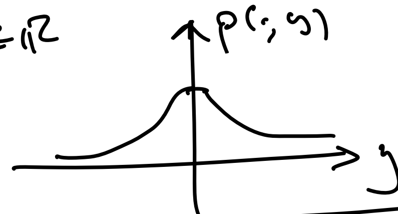
$\boxed{\text{Exemple 2}}$

$$\boxed{\begin{array}{l} X \sim N(0,1) \\ Y \sim N(0,1) \\ X \text{ et } Y \text{ ind} \\ \text{Quelle est la loi de } \begin{pmatrix} R \\ \theta \end{pmatrix} \text{ ?} \end{array} \qquad \left\{\begin{array}{l} X = R\cos\theta \\ Y = R\sin\theta \end{array}\right.}$$

$$p(x,\cdot) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad x \in \mathbb{R}$$



$$p(\cdot,y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \qquad y \in \mathbb{R}$$



X et Y ind $\Rightarrow$ $\boxed{p(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \quad (x,y) \in \mathbb{R}^2}$

La densité de (R,θ) est (application $\underline{\text{bijective}}$)

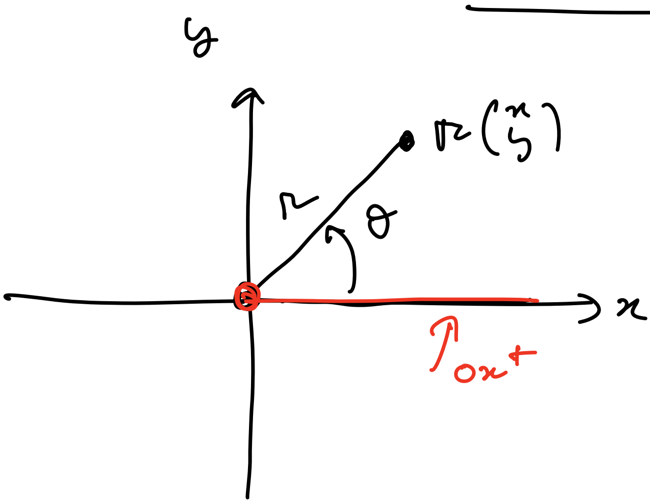
$$\pi(r,\theta) = \frac{1}{2\pi} \exp -\frac{1}{2}\left(r^2\cos^2\theta + r^2\sin^2\theta\right) \times |\det(J)|$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

donc $\det(J) = r \Rightarrow |\det(J)| = |r| = r$

donc $$\boxed{\pi(r,\theta) = \frac{r}{2\pi}\exp\left(-\frac{r^2}{2}\right) \quad (r,\theta) \in \, ?}$$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\left|\begin{array}{l} r = 0 \\ \theta \text{ quelconque} \end{array}\right. \Rightarrow \left|\begin{array}{l} x = 0 \\ y = 0 \end{array}\right.$$

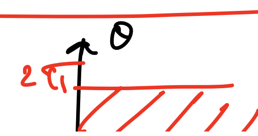
Le changement de variable est bijectif de

$\mathbb{R}^2 \setminus \{(0,0)\}$ dans $]0,+\infty[\times [0, 2\pi[$

Il est aussi bijectif de $\mathbb{R}^2 \setminus Ox^+$ dans

$]0,+\infty[\times]0, 2\pi[$

$$\boxed{\pi(r,\theta) = \frac{r}{2\pi} e^{-\frac{r^2}{2}} \quad (r,\theta) \in \,]0,+\infty[\times]0, 2\pi[}$$



Loi marginale de θ

$$\Pi(\cdot,\theta) = \int_0^{+\infty} \frac{r}{2\pi} e^{-r^2/2}\, dr$$

$$= \frac{1}{2\pi}\left[-e^{-r^2/2}\right]_0^{+\infty}$$

$$= \begin{cases} \dfrac{1}{2\pi} & \theta \in \left]0, 2\pi\right[\\ 0 & \text{sinon} \end{cases}$$

$$\boxed{\theta \sim \mathcal{U}(\left]0,2\pi\right[)}$$

Fonction caractéristique

Loi de $U = 2X + 3Y$?

$$\phi_U(t) = E\left[e^{iUt}\right] = E\left[e^{i(2X+3Y)t}\right]$$

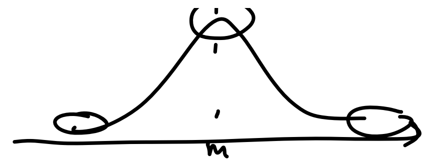
$$= E\left[e^{i2Xt}\, e^{3iYt}\right]$$

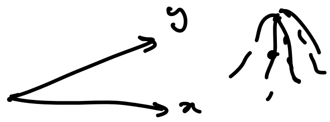
$$\underset{X \text{ et } Y \text{ ind}}{=} E\left[e^{i2Xt}\right] E\left[e^{3iYt}\right]$$

$$= \boxed{\phi_X(2t)\, \phi_Y(3t)}$$

Loi normale à n dimensions.

$n=1$ $\qquad p(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\dfrac{(x-m)^2}{2\sigma^2}\right] \quad x \in \mathbb{R}$





$$\Sigma = \begin{pmatrix} \mathrm{Var}\,X = \sigma_X^2 & \mathrm{cov}(X,Y) \\ & \mathrm{Var}\,Y = \sigma_Y^2 \end{pmatrix} \quad n=2$$

matrice de covariance

$$\Sigma = (\mathrm{Var}\,X) = (\sigma_X^2) \quad n=1$$

Rq

$$\Sigma = \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{pmatrix} \qquad \det(\Sigma) = \sigma_1^2 \times \dots \times \sigma_n^2$$

$$\sqrt{\det(\Sigma)} = \sqrt{\sigma_1^2 \dots \sigma_n^2}$$

$$\Sigma^{-1} = \begin{pmatrix} 1/\sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_n^2 \end{pmatrix}$$

$$\underline{(x_1 - m_1, \dots, x_n - m_n)} \; \underbrace{\Sigma^{-1} \begin{pmatrix} x_1 - m_1 \\ \vdots \\ x_n - m_n \end{pmatrix}}_{\begin{pmatrix} \frac{x_1 - m_1}{\sigma_1^2} \\ \vdots \\ \frac{x_n - m_n}{\sigma_n^2} \end{pmatrix}}$$

$$\exp\left[-\frac{1}{2} \left\{ \frac{(x_1 - m_1)^2}{\sigma_1^2} + \dots + \frac{(x_n - m_n)^2}{\sigma_n^2} \right\} \right] = \prod_{i=1}^{n} e^{-\frac{1}{2}\frac{(x_i - m_i)^2}{\sigma_i^2}}$$

$$p(x) = \frac{1}{(2\pi)^{n/2}\sqrt{\sigma_1^2 \dots \sigma_n^2}} \prod_{i=1}^{n} e^{-\frac{1}{2}\frac{(x_i - m_i)^2}{\sigma_i^2}}$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp - \frac{(x_i - m_i)^2}{2\sigma_i^2}$$

$\boxed{\text{TD du } 2/10/2023}$

$X \sim N(\underset{\text{var } X}{\overset{E(X)}{0}},1)$ donc $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\quad x \in \mathbb{R}$

$Y \begin{cases} \nearrow 1 & p \\ \searrow -1 & q = 1 \end{cases}$ $\qquad Z = XY$

Il est clair que Z est une variable continue à valeurs dans $\mathbb{R}$.
Donc il faut déterminer sa densité de probabilité $\pi(z)$.

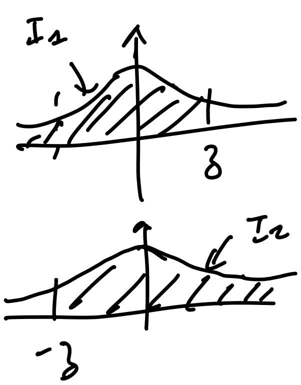
Méthode 1

$$P[Z \in]z, z+dz[] \simeq \pi(z)dz$$

Méthode 2 (Fonction de répartition)

$$P[Z < z] = P(XY < z)$$

$$P[Z<z] = P(XY<z)$$
$$= P[\, X<z \text{ et } Y=1 \text{ ou } -X<z \text{ et } Y=-1]$$
$$= P[\, X<z \text{ et } Y=1] + P(-X<z \text{ et } Y=-1)$$
$$\underset{X \text{ et } Y \text{ i.i}}{=} P(X<z)\underbrace{P(Y=1)}_{1/2} + P(X>-z)\underbrace{P(Y=-1)}_{1/2}$$
$$= \underbrace{\frac{1}{2}\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx}_{I_1} + \underbrace{\frac{1}{2}\int_{-z}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx}_{I_2}$$
$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$



par symétrie $I_1 = I_2$

$$\frac{dF(z)}{dz} = \pi(z) = \boxed{\frac{1}{\sqrt{2\pi}} e^{-z^2/2}}$$

donc $\boxed{Z \sim N(0,1)}$

Méthode 1

$$P[Z \in]z, z+dz[\,] \approx \pi(z)\,dz$$

$$\parallel$$

$$P[XY \in]z, z+dz[\,] = P[\, X \in]z, z+dz[\,, Y=1 \text{ ou } -X \in]z, z+dz[\,, Y=-1]$$

$$= \underbrace{P(Y=1)}_{1/2}\ \underbrace{P[X \in]z, z+dz[\,]}_{p(z)\,dz = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz}$$

$$+ \underbrace{P(Y=-1)}_{1/2}\ \underbrace{P[X \in]-z-dz, -z[\,]}_{\int_{-z-dz}^{-z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \approx \frac{1}{\sqrt{2\pi}} e^{-z^2/2}}$$

$$\underset{dz \text{ petit}}{\approx} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

donc $\boxed{Z \sim N(\underbrace{0}_{E(Z)}, \underbrace{1}_{Var[Z]})}$

2) $\mathrm{cov}(X, Z)$

3) $P[X+Z=0]$

$$\mathrm{cov}(X,Z) = E(XZ) - \underbrace{E(X)}_{0}\, \underbrace{E[Z]}_{0}$$

$$\underset{Z=XY}{=} E[YX^2] \underset{X \text{ et } Y \text{ ind}}{=} E(Y)\, \underbrace{E(X^2)}_{1}$$

$$Var\, X = E(X^2) - \underbrace{E^2[X]}_{0}$$

$$Y \begin{cases} 1 & 1/2 \\ -1 & 1/2 \end{cases}$$

$$E(Y) = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = \boxed{0}$$

donc $\boxed{\mathrm{cov}(X,Z) = 0}$

$$E(XZ) \underset{\text{espérance conditionnelle}}{=} E[E(XZ \mid X)] = E[X\, \underbrace{E[Z]}_{0}] = 0$$

$$= E\big[\underbrace{E[XZ|Z]}_{Z\underbrace{E[X]}_{0}}\big] = E[Z \times \underbrace{0}_{0}] = 0$$

3) $P[X+Z=0] = P(X + XY = 0)$

$= P[X(1+Y)=0]$

$X \sim N(0,1)$ donc $P(X=0)=0$!!!

$= P(1+Y=0) = \boxed{\frac{1}{2}}$

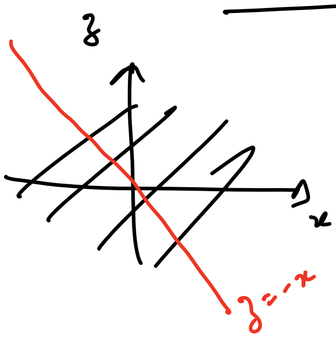
donc il y a une lien entre X et Z $\Rightarrow$ $\boxed{X \text{ et } Z \text{ non indépendants}}$

Rq : si X et Z étaient indépendants, alors $\pi(x,z) = \pi(x)\pi(z)$

$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$(x,z) \in \mathbb{R}^2$

$P(X+Z=0) = P[(X,Z) \in D,\ D = \{(x,z)/x+z=0\}]$

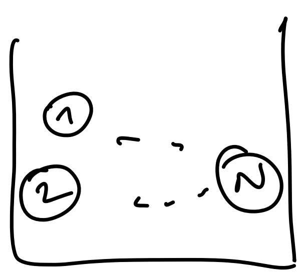


$= \iint_D \frac{1}{2\pi} e^{-\frac{x^2+z^2}{2}} dx dz = \boxed{0}$

X et Z ind $\Rightarrow P[X+Z=0] = 0$

Contraposé

$P(X+Z=0) \neq 0 \Rightarrow X$ et Z non ind



2 tirages sans remise

X_1 = première boule

X_2 = seconde boule

1) lois de X_1, X_2 et de (X_1, X_2)

2) $\text{cov}(X_1, X_2)$

Loi de X_1

X_1 est une va discrète à valeurs dans $\{1, \dots, N\}$ avec

$$P[X_1 = k] = \frac{1}{N} \quad k \in \{1, \dots, N\}$$

Loi de X_2

X_2 est une va discrète à valeurs dans $\{1, \dots, N\}$ avec

$$P[X_2 = i] = P\big[(X_1 = 1, X_2 = i) \text{ ou } \dots (X_1 = i-1, X_2 = i) \text{ ou } \widehat{(X_1 = i, X_2 = i)}$$
$$\text{ou } (X_1 = i+1, X_2 = i) \text{ ou } \dots (X_1 = N, X_2 = i)\big]$$

$$= \sum_{\substack{k \neq i \\ k = 1, \dots, N}} \underbrace{P[X_1 = k, X_2 = i]}_{P[X_2 = i \mid X_1 = k]\, P(X_1 = k)}$$

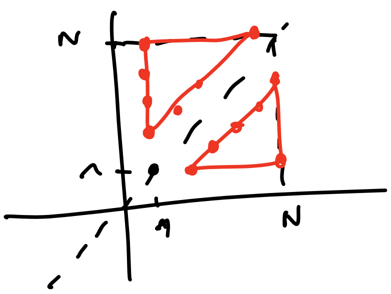
$$P[X_2 = i \mid X_1 = k] = \frac{1}{N-1}, \quad P(X_1 = k) = \frac{1}{N}$$

$$= (N-1) \times \frac{1}{N-1} \times \frac{1}{N} = \frac{1}{N}$$

X_1 et X_2 suivent des lois uniformes sur $\{1, \dots, N\}$.

Loi de (X_1, X_2)

(X_1, X_2) va à valeurs dans $\left\{(i, j),\ \begin{matrix} i = 1, \dots, N \\ j = 1, \dots, N \end{matrix},\ i \neq j\right\}$



$$P[X_1 = i, X_2 = j] = \begin{cases} 0 & i = j \\ \underbrace{P(X_2 = j \mid X_1 = i)}_{\frac{1}{N-1}}\ \underbrace{P(X_1 = i)}_{\frac{1}{N}} & i \neq j \end{cases}$$

$$= \begin{cases} 0 & i = j \end{cases}$$

$$\boxed{\quad \frac{1}{N(N-1)} \quad i \neq j}$$

2) Covariance de (X_1, X_2)

$$\operatorname{cov}(X_1, X_2) = E[X_1 X_2] - E(X_1)E(X_2)$$

$$X_1 \begin{cases} 1 & \frac{1}{N} \\ N & \frac{1}{N} \end{cases}$$

$$E(X_1) = 1 \times \frac{1}{N} + \dots N \times \frac{1}{N} = \frac{1}{N} \frac{N(N+1)}{2} = \boxed{\frac{N+1}{2}}$$

$$E(X_2) = \boxed{\frac{N+1}{2}}$$

$$E[X_1 X_2] = \sum_i \sum_j x_i y_j \, P[X_1 = x_i, X_2 = y_j]$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} i j \, \underbrace{P[X_1 = i, X_2 = j]}_{\begin{cases} 0 & i = j \\ \frac{1}{N(N-1)} & i \neq j \end{cases}}$$

$$= \sum\sum_{i \neq j} \frac{ij}{N(N-1)} = \frac{1}{N(N-1)} \underbrace{\sum\sum_{i \neq j} ij}_{\sum_{i,j} ij - \sum_{i=1}^{N} i^2}$$

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\overbrace{(1+2 \dots + N)}^{\Sigma i} \overbrace{(1 + \dots + N)}^{\Sigma j}$$

$$\sum_{i,j} ij = (\Sigma i)(\Sigma j)$$
$$= \left[\frac{N(N+1)}{2}\right]^2$$

$$E(X_1 X_2) = \frac{1}{\cancel{N}(N-1)} \left[\frac{N^{\cancel{2}}(N+1)^2}{4} - \frac{\cancel{N}(N+1)(2N+1)}{6} \right]$$

$$= \frac{N+1}{N-1} \left[\frac{N(N+1)}{4} - \frac{2N+1}{6} \right]$$

$$\frac{3N^2+3N-4N-2}{12} = \frac{3N^2-N-2}{12} = \frac{(N-1)(3N+2)}{12}$$

$$= \boxed{\frac{(N+1)(3N+2)}{12}}$$

$$\mathrm{cov}(X_1, X_2) = \frac{(N+1)(3N+2)}{12} - \left(\frac{N(N+1)}{2}\right)^2$$

$$= \cdots$$

Cours du 9/10/2023

$n=1$ $\boxed{\Sigma = (\sigma^2)}$ $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$ $x \in \mathbb{R}^2$

n $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $m = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$ $\overset{n}{\longleftrightarrow} \Sigma \quad \updownarrow n$

$$\boxed{X \sim N_n(m, \Sigma)}$$

Σ symétrique définie positive

$$\boxed{\begin{aligned} &\Sigma^T = \Sigma \\ &\Sigma \text{ positive} \iff x^T \Sigma x \geq 0 \quad \forall x \in \mathbb{R}^n \\ &\Sigma \text{ définie} \iff x^T \Sigma x = 0 \implies x = 0 \end{aligned}}$$

On va voir que $m = \begin{pmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{pmatrix}$ et $\Sigma = \begin{pmatrix} \mathrm{Var} X_1 & & \\ & \ddots & \\ \boxed{\mathrm{cov}(X_i, X_j)} & & \mathrm{Var} X_n \end{pmatrix}$

i, j

$$\phi(u) = E\left[e^{iu^T X}\right] = E\left[e^{i(u_1 x_1 + \cdots + u_n x_n)^2}\right]$$

$$= \iint \cdots \int e^{i(u x_1 + \dots + u_n x_n)} p(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

$$= \dots =$$

$$\Phi(u) = E\left[e^{u^T x}\right] = E\left[e^{u_1 x_1 + \dots + u_n x_n}\right]$$

$$\frac{\partial \Phi}{\partial u_1} = E\left[e^{u^T x} x_1\right]$$

$$\boxed{\left.\frac{\partial \Phi}{\partial u_1}\right|_{u=0} = E(X_1)}$$

$$\left.\frac{\partial^2 \Phi}{\partial u_1 \partial u_2}\right|_{u=0} = E\left[e^{u^T x} x_1 x_2\right]\Big|_{u=0}$$

$$= E\left[x_1 x_2\right]$$

Conclusion: si $X \sim N\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}5 & 1 & 2\\ 1 & 4 & 3\\ 2 & 3 & 8\end{pmatrix}\right)$

$\Rightarrow$

$E(X_1) = 1 \quad E(X_2) = 2 \quad E(X_3) = 3$

$Var(X_1) = 5 \quad Var(X_2) = 4 \quad Var X_3 = 8$

$cov(X_1, X_2) = 1 \quad cov(X_1, X_3) = 2 \quad cov(X_2, X_3) = 3$

Rq: $n = 1$ $\quad p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\underbrace{-\frac{(x-m)^2}{2\sigma^2}}_{-q(x)}\right]$

$$\boxed{\begin{aligned} q'(x) &= +\frac{1}{2\sigma^2} 2(x-m) = \frac{x-m}{\sigma^2} \\ q''(x) &= \frac{1}{\sigma^2} \end{aligned}}$$

$$\exp\left(-2x^2 + x - 1\right) \qquad q(x) = 2x^2 - x + 1$$

$$q'(x) = 4x - 1$$

$$q''(x) = 4 \Rightarrow \boxed{\sigma^2 = \frac{1}{4}}$$

pour avoir m $\qquad 4x - 1 = \dfrac{x-m}{\sigma^2} \underset{\sigma^2 = \frac{1}{4}}{\overset{\uparrow}{=}} 4(x-m)$

$$-1 = -4m \Rightarrow \boxed{m = \frac{1}{4}}$$

en dimension

$x \in \mathbb{R}^n$

$$q'(x) = \begin{bmatrix} \dfrac{\partial q}{\partial x_1} \\ \vdots \\ \dfrac{\partial q}{\partial x_n} \end{bmatrix}$$

$$q''(x) = \begin{bmatrix} \dfrac{\partial^2 q}{\partial x_1^2} & \dfrac{\partial^2 q}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 q}{\partial x_1 \partial x_n} \\ & \ddots & & \\ & & & \dfrac{\partial^2 q}{\partial x_n^2} \end{bmatrix}$$

On montre que $\boxed{q''(x) = \Sigma^{-1}}$ $\qquad q(x) = \dfrac{(x-m)^T \Sigma^{-1} (x-m)}{2}$

$$\boxed{q'(x) = \Sigma^{-1}(x-m)}$$

$$p(x,y) \propto \exp\left(-x^2 - \frac{3}{2}y^2 - xy + 4x + 7y\right) \qquad m\,? \quad \Sigma\,?$$

$$q(x,y) = x^2 + \frac{3}{2}y^2 + xy - 4x - 7y$$

$$q'' = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = \Sigma^{-1}$$

$$\boxed{\Sigma = \begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}^{-1}} = \frac{1}{5}\begin{pmatrix}3 & -1\\ -1 & 2\end{pmatrix}$$

$$q'(x) = \begin{pmatrix} \dfrac{\partial q}{\partial x} = 2x + y - 4 \\[2ex] \dfrac{\partial q}{\partial y} = 3y + x - 7 \end{pmatrix} = \Sigma^{-1}\begin{pmatrix}x - m_1\\ y - m_2\end{pmatrix} = \begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}x - m_1\\ y - m_2\end{pmatrix}$$

$$\begin{cases} 2\cancel{x} + \cancel{y} - 4 = 2(\cancel{x} - m_1) + \cancel{y} - m_2 \\ 3\cancel{y} + \cancel{x} - 7 = \cancel{x} - m_1 + 3(\cancel{y} - m_2) \end{cases}$$

d'où $\begin{cases} -4 = -2m_1 - m_2 \\ -7 = -m_1 - 3m_2 \end{cases} \Rightarrow \begin{cases} m_1 = 1 \\ m_2 = 2 \end{cases}$

Transformation affine $Y = AX + B$

$$X = \begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

$$A = \underbrace{\left(\qquad\qquad\right)}_{p \times n}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

$p \leq n$

$$\phi_Y(u) = E\left[e^{iu^T Y}\right]$$

$$\overset{Y = AX + B}{=} E\left[e^{iu^T(AX+B)}\right] \qquad e^{iu^T AX}\ \overbrace{e^{iu^T B}}^{\text{constante}}$$

$$= e^{iu^T B}\, E\left[e^{\overbrace{iu^T A}^{v^T} X}\right]$$

$v^T = u^T A \iff \boxed{v = A^T u}$

$$= e^{iu^T B}\, \phi_X(v)$$

$$= e^{iu^T B} \, \phi_X(A^T u)$$

$$X \sim N_n(m, \Sigma) \Rightarrow \phi_X(u) = \exp\left(i u^T m - \frac{1}{2} u^T \Sigma u\right)$$

donc $\phi_Y(u) = e^{iu^T B} \exp\left(i u^T A m - \frac{1}{2} u^T A \Sigma A^T u\right)$

$$= \exp\left[i \, u^T \underbrace{(B + Am)}_{m_Y} - \frac{1}{2} u^T \underbrace{(A \Sigma A^T)}_{\Sigma_Y} u \right]$$

donc $Y \sim N(Am + B, A\Sigma A^T)$ à la condition d'avoir $A\Sigma A^T = M$ matrice symétrique définie positive

M symétrique ? OUI $\qquad M^T = (A\Sigma A^T)^T = A \Sigma^T A^T = A \Sigma A^T = M$

car Σ est la matrice de covariance d'un vecteur gaussien

M positive ? OUI $\qquad x^T M x = \underbrace{x^T A}_{y^T} \Sigma \underbrace{A^T x}_{y} = y^T \Sigma y \geq 0$

car Σ est positive

M définie $\qquad x^T M x = 0 \Rightarrow y^T \Sigma y = 0$ avec $y = A^T x$

$\Rightarrow y = A^T x = 0$ car Σ définie

il faut $\boxed{A^T x = 0 \Rightarrow x = 0}$

il faut que A soit de rang p !!

$$X \sim N\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{pmatrix} \right)$$

Quelle est la loi de $Y = \big(2X_1 + X_2 - 3 \big)$?

$$x_1 - x_3 + 1$$

$Y = AX + B$ avec $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ $B = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\mathrm{rg}\, A = 2$

si $A = \begin{pmatrix} 1 & 1 & 2 \\ -2 & -2 & -4 \end{pmatrix}$ $\mathrm{rg}\, A = 1$

donc $Y \sim N(Am + B,\ A\Sigma A^T)$

$$Am + B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A\Sigma A^T = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix} = \dots$$

Rq $x_1 + x_2 + x_3 = \underbrace{(1\ 1\ 1)}_{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Lois marginales $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \sim N(m, \Sigma)$ $m = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 7 \end{pmatrix}$

Ce sont les lois de x_1, x_2, x_3, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$ $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$

$$x_1 = \underbrace{(1\ 0\ 0)}_{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\mathrm{rg}\, A = 1 \Rightarrow x_1 \sim N(\overset{?}{1}, \overset{?}{4})$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

m_1

$$\operatorname{rg} A = 2 \Rightarrow \boxed{\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N\Big(\underbrace{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}_{m_3}, \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix} \Big)}$$

Indépendance

$$X = \begin{pmatrix} X_1 \\ \hline X_2 \\ X_3 \end{pmatrix} \begin{matrix} X' \\ X'' \end{matrix}$$

X_1 ind de $\begin{pmatrix} X_2 \\ X_3 \end{pmatrix}$? Cas 1

X_2 ind de $\begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$? Cas 2

$$\begin{pmatrix} X_2 \\ \hline X_1 \\ X_3 \end{pmatrix} \begin{matrix} X' \\ X'' \end{matrix}$$

Cas 1 si X_1 ind de $\begin{pmatrix} X_2 \\ X_3 \end{pmatrix}$ alors

X_1 ind de X_2 $\Rightarrow \operatorname{cov}(X_1, X_2) = 0$
et
X_1 ind de X_3 $\Rightarrow \operatorname{cov}(X_1, X_3) = 0$

donc $\Sigma = \left(\begin{array}{c|cc} \sigma_1^2 & 0 & 0 \\ \hline 0 & \sigma_2^2 & a \\ 0 & a & \sigma_3^2 \end{array}\right)$

$$(x-m)^T \Sigma^{-1} (x-m) = (x_1 - m_1, x_2 - m_2, x_3 - m_3) \left(\begin{array}{c|c} \frac{1}{\sigma_1^2} & 0 \quad 0 \\ \hline 0 & \begin{pmatrix} \sigma_2^2 & a \\ a & \sigma_3^2 \end{pmatrix}^{-1} \\ 0 & \end{array}\right) \begin{pmatrix} x_1 - m_1 \\ x_2 - m_2 \\ x_3 - m_3 \end{pmatrix}$$

$$= (x_1 - m_1, x_2 - m_2, x_3 - m_3) \begin{pmatrix} \frac{x_1 - m_1}{\sigma_1^2} \\ \begin{pmatrix} \sigma_2^2 & a \\ a & \sigma_3^2 \end{pmatrix}^{-1} \begin{pmatrix} x_2 - m_2 \\ x_3 - m_3 \end{pmatrix} \end{pmatrix}$$

$$= \frac{(x_1 - m_1)^2}{\sigma_1^2} + (x_2 - m_2, x_3 - m_3) \begin{pmatrix} \sigma_2^2 & a \\ a & \sigma_3^2 \end{pmatrix} \begin{pmatrix} x_2 - m_2 \\ x_3 - m_3 \end{pmatrix}$$

$$\exp\left[-\frac{1}{2}(x-m)^T \Sigma^{-1} (x-m)\right] = \exp\left[-\frac{1}{2\sigma_1^2}(x_1 - m_1)^2\right] \times \exp\left[-\frac{1}{2}(x'' - m'')(\Sigma'')^{-1}(x'' - m'')\right]$$

$\Rightarrow p(x) = p(x', \cdot) \times p(\cdot, x'') \Rightarrow \boxed{x' \text{ et } x'' \text{ ind}}$

Application : $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/4 \\ 1/2 & 1/4 & 1 \end{pmatrix} \right)$

$\mathrm{cov}(X_1, X_2) = 0 \Rightarrow X_1$ et X_2 ind

$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ vect gaussien

TD4 du 9/10/2023

$$\begin{array}{c|c} 0 & k e^{-\frac{x^2+y^2}{2}} \\ \hline k e^{-\frac{x^2+y^2}{2}} & 0 \end{array}$$

Rappel $\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 \Rightarrow \int_{\mathbb{R}} e^{-x^2/2} dx = \sqrt{2\pi}$

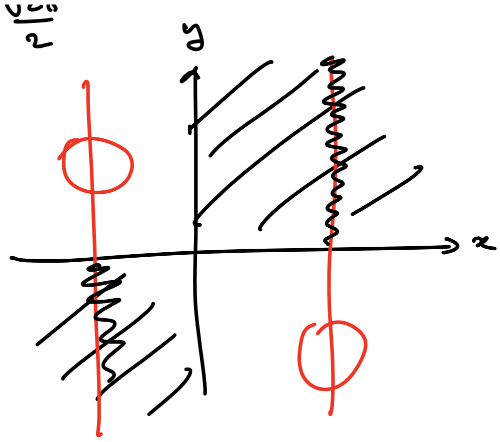
i) $\iint_{\mathbb{R}^2} f(x,y)\, dx\, dy = 1 \Rightarrow \iint_{(\mathbb{R}^+)^2 \cup (\mathbb{R}^-)^2} k e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}} dx\, dy = 1$

$$\Rightarrow 1 = k \left[\int_0^{+\infty} \int_0^{+\infty} e^{-x^2/2} e^{-y^2/2} dx\, dy + \int_{-\infty}^0 \int_{-\infty}^0 e^{-x^2/2} e^{-y^2/2} dx\, dy \right]$$

$$1 = 2k \underbrace{\left(\int_0^{+\infty} e^{-x^2/2} dx \right)}\left(\int_0^{+\infty} e^{-y^2/2} dy \right) \qquad \boxed{k = \frac{1}{\pi}}$$

$\sqrt{2\pi}$

$\frac{\sqrt{2\pi}}{2}$ $\frac{\sqrt{2\pi}}{2}$

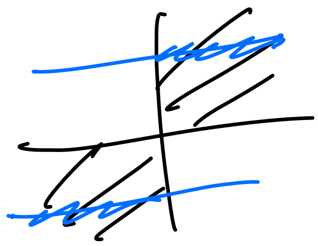


2) Lois marginales de X et de Y

$$f(x,\cdot) = \int_{\mathbb{R}} f(x,y)\,dy$$

$$= \begin{cases} x \geq 0 & \int_0^{+\infty} \frac{1}{\pi} e^{-x^2/2} e^{-y^2/2}\,dy \\ x \leq 0 & \int_{-\infty}^0 \frac{1}{\pi} e^{-x^2/2} e^{-y^2/2}\,dy \end{cases}$$

donc $f(x,\cdot) = \begin{cases} \frac{1}{\pi} e^{-x^2/2} \frac{\sqrt{2\pi}}{2} & x \geq 0 \\ \frac{1}{\pi} e^{-x^2/2} \frac{\sqrt{2\pi}}{2} & x \leq 0 \end{cases}$



donc $\boxed{f(x,\cdot) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad x \in \mathbb{R}}$

$X \sim N(0,1)$ donc $E[X] = 0$
$Var[X] = 1$

De même

$$f(\cdot,y) = \begin{cases} y \geq 0 & \int_0^{+\infty} \frac{1}{\pi} e^{-x^2/2} e^{-y^2/2}\,dx \\ y \leq 0 & \int_0^{+\infty} \frac{1}{\pi} e^{-x^2/2} e^{-y^2/2}\,dx \end{cases}$$

m̂ calculs

$Y \sim N(0,1)$

Remarque $\boxed{X \sim N(0,1),\ Y \sim N(0,1) \text{ et pourtant } \binom{X}{Y} \text{ n'est pas un vecteur gaussien ?!}}$

3) $cov(X,Y) = ?$ Rq : On sait que X et Y ne sont pas indépendantes car le domaine de définition n'est pas $\mathbb{R}^2$!!

$$Cov(X,Y) = E(XY) - \underbrace{E(X)}_{0}\underbrace{E(Y)}_{0} = E[XY]$$

$$E(XY) = \int_0^{+\infty}\int_0^{+\infty} + \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{xy}{\pi} e^{-\frac{x^2}{2}} e^{-y^2/2}\, dx\, dy$$

$$= \frac{2}{\pi} \underbrace{\left(\int_0^{+\infty} x e^{-x^2/2} dx\right)}_{\left[-e^{-x^2/2}\right]_0^{+\infty} = 1} \underbrace{\left(\int_0^{+\infty} y e^{-y^2/2} dy\right)}_{=1} = \boxed{\frac{2}{\pi}}$$

4) Lois de $Z = X+Y$ et $U = X-Y$?

Espérance conditionnelle

$$E(XY) = E[\underbrace{E[XY|X]}]$$

loi de $Y|X$

$$\frac{f(x,y)}{f(x)} = \frac{\frac{1}{\pi} e^{-\frac{\cancel{x^2}+y^2}{2}}}{\frac{1}{\sqrt{2\pi}} \cancel{e^{-x^2/2}}}$$

$(x,y) \in (\mathbb{R}^+)^2 \cup (\mathbb{R}^-)^2$

donc

$$f(y|x) = \sqrt{\frac{2}{\pi}}\, e^{-y^2/2}$$

si $x>0$ $\quad f(y|x) = \sqrt{\frac{2}{\pi}} e^{-y^2/2}$ $\quad y>0$

si $x<0$ $\quad f(y|x) = \sqrt{\frac{2}{\pi}} e^{-y^2/2}$ $\quad y<0$

donc si $x>0$

$$E(Y|X) = \int_0^{+\infty} \sqrt{\frac{2}{\pi}}\, y\, e^{-y^2/2} dy$$

$$= \sqrt{\frac{2}{\pi}} \left[-e^{-y^2/2}\right]_0^{+\infty}$$

$$= \sqrt{\frac{2}{\pi}} \quad x>0$$

donc $E(XY) = E[XE(Y|X)]$

$$= \int_0^{+\infty} x \cdot \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$+ \int_{-\infty}^{0} x \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= \sqrt{\frac{\cancel{2}}{\pi}} \frac{1}{\sqrt{\cancel{2}\pi}} \underbrace{\left[-e^{-x^2/2}\right]_0^{+\infty}}_{1}$$

$$+ \sqrt{\frac{\cancel{2}}{\pi}} \frac{1}{\sqrt{\cancel{2}\pi}}$$

$$= \boxed{\frac{2}{\pi}}$$

$\begin{cases} x>0 \\ y>0 \end{cases}$ ou $\begin{cases} x<0 \\ y<0 \end{cases}$

$$\begin{cases} Z = X+Y \\ U = X-Y \end{cases} \Leftrightarrow \boxed{\begin{cases} x = \frac{1}{2}(z+u) \\ y = \frac{1}{2}(z-u) \end{cases}}$$

donc le changement de variable est bijectif

Jacobien

$$\begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\boxed{|J| = \frac{1}{2}}$$

Densité de (Z,U)

$$\pi(z,u) = \frac{1}{\pi} \exp\left\{-\frac{1}{2}\left[\left(\frac{z+u}{2}\right)^2 + \left(\frac{z-u}{2}\right)^2\right]\right\} \times \frac{1}{2}$$

donc $\boxed{\pi(z,u) = \frac{1}{2\pi} \exp\left(-\frac{z^2+u^2}{4}\right)}$ $(z,u) \in ?$

Domaine de (Z,U)

Domaine de (X,Y)

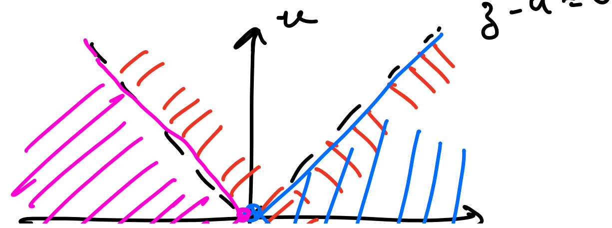
$\begin{cases} x>0 \\ y>0 \end{cases} \Leftrightarrow \boxed{\begin{cases} z+u>0 \\ z-u>0 \end{cases}}$

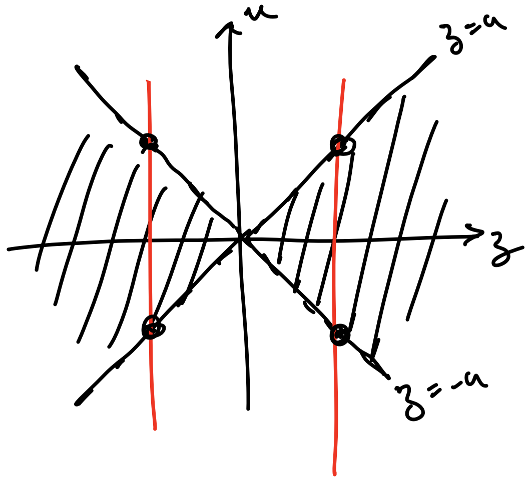
ou

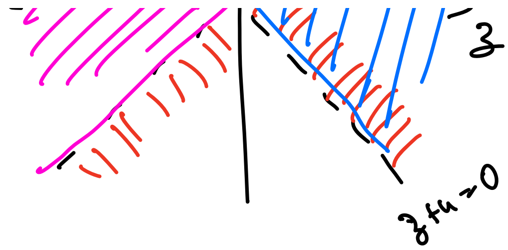
$\begin{cases} x<0 \\ y<0 \end{cases} \Leftrightarrow \boxed{\begin{cases} z+u<0 \\ z-u<0 \end{cases}}$

$X = \frac{Z+U}{2}$

$Y = \frac{Z-U}{2}$







Loi de Z

$$\pi(z,\cdot) = \begin{cases} z\geq 0 & \displaystyle\int_{-z}^{z} \frac{1}{2\pi} e^{-\frac{z^2+u^2}{4}} \, du \\ z<0 & \displaystyle\int_{z}^{-z} \frac{1}{2\pi} e^{-\frac{z^2+u^2}{4}} \, du \end{cases}$$

Dans les 2 cas

$$\pi(z,\cdot) = \int_{-|z|}^{|z|} \frac{1}{2\pi} e^{-\frac{z^2+u^2}{4}} \, du$$

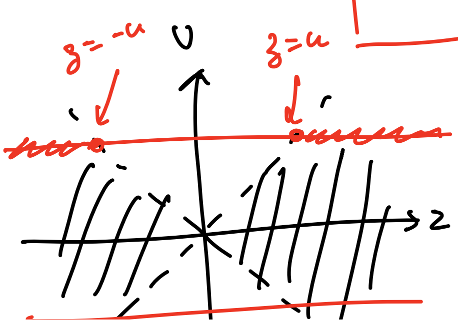
$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \int_{-|z|}^{|z|} \frac{1}{\sqrt{2\pi}} e^{-u^2/4} \, du$$

$$\boxed{\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du = F(x)}$$

changement de variable $v = \frac{u}{\sqrt{2}}$

$$\pi(z,\cdot) = \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \int_{-\frac{|z|}{\sqrt{2}}}^{\frac{|z|}{\sqrt{2}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \sqrt{2} \, dv$$

$$\boxed{\pi(z,\cdot) = \frac{1}{\sqrt{\pi}} e^{-z^2/4} \left[F\left(\frac{|z|}{\sqrt{2}}\right) - F\left(-\frac{|z|}{\sqrt{2}}\right)\right]}$$

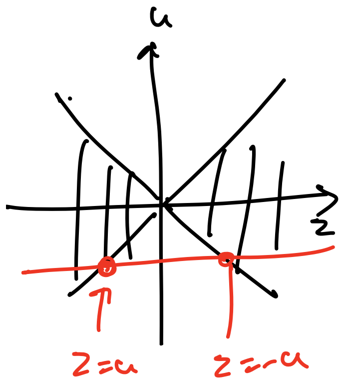


Loi de U

si $u \geq 0$ $\quad \pi(\cdot,u) = \displaystyle\int_{-\infty}^{-u} + \int_{u}^{+\infty} \frac{1}{2\pi} \exp\left(-\frac{z^2+u^2}{4}\right) dz$

$$\pi(\cdot,u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/4} \left(\int_{-\infty}^{-u} + \int_{u}^{+\infty} \right) \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \, dz$$

$v = \frac{z}{\sqrt{2}} \qquad \frac{v^2}{2} = \frac{z^2}{4}$



$$\pi(\cdot,u) = \frac{1}{\sqrt{\cancel{2}\pi}} e^{-u^2/4} \left(\int_{-\infty}^{-\frac{u}{\sqrt{2}}} + \int_{\frac{u}{\sqrt{2}}}^{+\infty} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \cancel{\sqrt{2}} \, dv$$

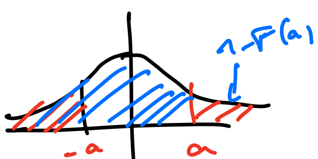
donc $\boxed{\pi(\cdot,u) = \frac{1}{\sqrt{\pi}} e^{-u^2/4} \left[F\left(-\frac{u}{\sqrt{2}}\right) + 1 - F\left(\frac{u}{\sqrt{2}}\right) \right] \quad u>0}$

si $u<0$

$$\pi(\cdot,u) = \left(\int_{-\infty}^{u} + \int_{-u}^{+\infty} \right) \frac{1}{2\pi} \exp\left(-\frac{z^2+u^2}{4}\right) dz$$

on change u en -u

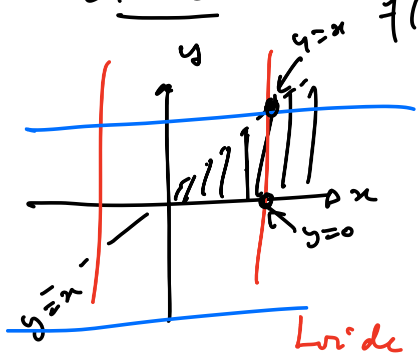
$$\boxed{\pi(\cdot,u) = \frac{1}{\sqrt{\pi}} e^{-u^2/4} \left(F\left(-\frac{|u|}{\sqrt{2}}\right) + 1 - F\left(\frac{|u|}{\sqrt{2}}\right) \right) \quad \forall u \in \mathbb{R}}$$



$F(-a) = 1 - F(a)$

Exercice 3

$$f(x,y) = \begin{cases} \theta^2 e^{-\theta x} & 0<y<x \\ 0 & \text{sinon} \end{cases}$$



1) Lois marginales de X et de Y

2) Loi de $Z = \frac{Y}{X}$

Loi de X

$$f(x,\cdot) = \int_{\mathbb{R}} f(x,y)\,dy = \begin{cases} 0 & x \leq 0 \\ \int_0^x \theta^2 e^{-\theta x} \, dy & x>0 \end{cases}$$

donc $f(x,\cdot) = \{ \theta^2 x e^{-\theta x} \quad x>0$

$$\begin{array}{l|l} \ldots & 0 \quad \text{sinon} \end{array}$$

$$\boxed{X \sim \Gamma(2, \theta)}$$

loi de Y

$$f(\cdot, y) = \begin{cases} 0 & y < 0 \\ \displaystyle\int_y^{+\infty} \underbrace{\theta^2 e^{-\theta x}}_{\left[-\theta e^{-\theta x}\right]_y^{+\infty} = \theta e^{-\theta y}} dx & y \leq 0 \end{cases}$$

donc $f(\cdot, y) = \begin{cases} 0 & y \leq 0 \\ \theta e^{-\theta y} & y > 0 \end{cases}$

$$\boxed{Y \sim \Gamma(1, \theta)}$$

Cours du 16/10/2023

$$X_n \to 1 \qquad P(X_n = 1) = \frac{1}{n}$$

$$X_n \to 0 \qquad P(X_n = 0) = 1 - \frac{1}{n}$$

$$X_n \xrightarrow{\mathcal{L}} X = 0 \ ?$$

$$\phi_n(t) = E\left[e^{itX_n}\right] = \underbrace{e^{it \times 0}}_{1} P(X_n = 0) + e^{it \times 1} P(X_n = 1)$$

$$= \underbrace{1 - \frac{1}{n}}_{\to 1} + \underbrace{\frac{1}{n} e^{it}}_{\to 0}$$

$n \to \infty$

donc $\phi_n(t) \underset{n \to +\infty}{\longrightarrow} 1$

$$\phi(t) = E\left[e^{itx}\right] \underset{x=0}{=} E\left[e^{itx0}\right] = E(1) = 1$$

continue en $t=0$

$\phi_n(t) \underset{n \to +\infty}{\longrightarrow} \phi(t)$
ϕ continue en $t=0$

$\overset{\text{LEVY}}{\Longrightarrow}$ $\boxed{X_n \underset{n \to +\infty}{\overset{\mathcal{L}}{\longrightarrow}} X = 0}$

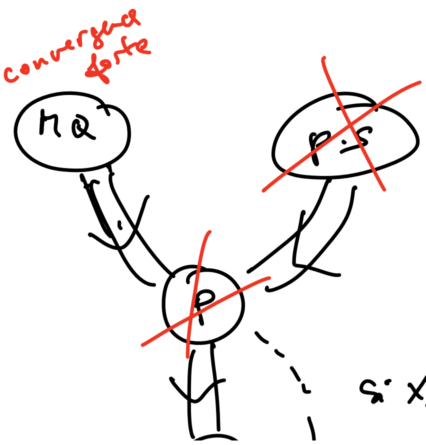
$$\boxed{\begin{array}{lll} X_n \begin{array}{l} \nearrow 1 \\ \searrow 0 \end{array} & \begin{array}{l} \frac{1}{n} \\ 1 - \frac{1}{n} \end{array} \\ X_n \underset{n \to +\infty}{\overset{M.Q}{\longrightarrow}} X = 0 \; ? \end{array}}$$

$$E\left[(X_n - x)^2\right] \underset{x=0}{=} E\left[X_n^2\right] = 1^2 \times \frac{1}{n} + 0^2 \times \left(1 - \frac{1}{n}\right)$$

$$= \frac{1}{n} \underset{n \to +\infty}{\longrightarrow} 0$$

donc $\boxed{X_n \underset{n \to +\infty}{\overset{MQ}{\longrightarrow}} X = 0}$

<u>Comparaison</u>



si $X_n \xrightarrow{\mathcal{L}} X$ = constante alors

(2) / $X_n \xrightarrow{P} X = \text{constante}$

convergence faible

$$E[(\bar{X}_n - m)^2] = E\left[\left(\frac{1}{n}\sum_{k=1}^{n} X_k - m\right)^2\right]$$

$\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$

$$= E\left[\left\{\frac{1}{n}\sum_{k=1}^{n}(X_k - m)\right\}^2\right]$$

$m = \frac{1}{n}\sum_{k=1}^{n} m$

$$= \frac{1}{n^2} E\left[\sum_{k=1}^{n}(X_k - m) \sum_{l=1}^{n}(X_l - m)\right]$$

$$\boxed{\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{l=1}^{n} b_l\right) = \sum_{k=1}^{n}\sum_{l=1}^{n} a_k b_l}$$

$$(a_1 + \dots + a_n)(b_1 + \dots + b_n) = a_1 b_1 + a_1 b_2 + \dots + a_n b_n$$

$$= \frac{1}{n^2} E\left[\sum_{k=1}^{n}\sum_{l=1}^{n}(X_k - m)(X_l - m)\right]$$

$$= \frac{1}{n^2}\sum_{k=1}^{n}\sum_{l=1}^{n} E[(X_k - m)(X_l - m)]$$

$$\begin{cases} \sigma^2 & k = l \\ 0 & k \neq l \end{cases}$$

si $k \neq l$ $\quad E[(X_k - m)(X_l - m)] = \text{cov}(X_k, X_l) = 0$

car X_k et X_l ind

donc $\quad E[(\bar{X}_n - m)^2] = \frac{1}{n^2} \times n\sigma^2 = \boxed{\frac{\sigma^2}{n}} \xrightarrow[n \to +\infty]{} 0$

Théorème de la limite centrale

$$\sum_{k=1}^{n} X_k \approx N(nm \,,\, n\sigma^2)$$

$$E\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} \underbrace{E(X_k)}_{m} = nm$$

$$\mathrm{Var}\left(\sum_{k=1}^{n} X_k\right) \underset{X_1,\dots,X_n \text{ ind}}{=} \sum_{k=1}^{n} \underbrace{\mathrm{Var}\, X_k}_{\sigma^2} = n\sigma^2$$

$$\frac{1}{n}\sum_{k=1}^{n} X_k \approx N\left(m \,,\, \frac{\sigma^2}{n}\right)$$

$$\mathrm{Var}\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \frac{1}{n^2}\mathrm{Var}\left(\sum_{k=1}^{n} X_k\right) \underset{X_1,\dots,X_n \text{ iid}}{=} \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Propriété : $\boxed{\text{Si } X \text{ et } Y \text{ sont des variables aléatoires indépendantes alors } \mathrm{Var}(X+Y) = \mathrm{Var}\, X + \mathrm{Var}\, Y}$

En effet

$$\mathrm{Var}(X+Y) = E\left((X+Y)^2\right) - \left(E[X+Y]\right)^2$$

$$= E\left(X^2 + Y^2 + 2XY\right) - \underbrace{\left(E(X) + E(Y)\right)^2}_{E^2[X] + E^2(Y) + 2E(X)E(Y)}$$

$$= \underbrace{E(X^2) - E^2[X]}_{\mathrm{Var}\, X} + \underbrace{E(Y^2) - E^2(Y)}_{\mathrm{Var}\, Y} + 2\left(E[XY] - E(X)E(Y)\right)$$

$$\boxed{\mathrm{Var}(X+Y) = \mathrm{Var}\,X + \mathrm{Var}\,Y + 2\,\mathrm{cov}(X,Y)}$$

Si X et Y ind $\Rightarrow$ $\mathrm{cov}(X,Y) = 0$ $\Rightarrow$ $\mathrm{Var}(X+Y) = \mathrm{Var}\,X + \mathrm{Var}\,Y$

CQFD

THE END!

TD du 16/10/2023

$X \sim N(0,1)$ $Y \sim N(0,1)$ $Z \sim N(0,1)$
X, Y et Z indépendants

1) Loi de $U = X+Y+Z$

Méthode 1

$$\varphi_U(t) = E[e^{itU}] = E[e^{it(X+Y+Z)}]$$
$$= E[e^{itX} e^{itY} e^{itZ}]$$
$$\overset{X,Y,Z \text{ ind}}{=} \phi_X(t)\,\phi_Y(t)\,\phi_Z(t)$$
$$= e^{-t^2/2} \times e^{-t^2/2} \times e^{-t^2/2}$$
$$= e^{-3t^2/2}$$

qui est la fonction caractéristique d'une loi normale $N(0,3)$

donc $\boxed{U \sim N(0,3)}$

Rq : $U = X+Y+Z$

Méthode 2

$$U = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$= A\,V + b \qquad V = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$

$b = (0)$

Rappel : $\boxed{\text{Si } V \sim N(m, \Sigma) \text{ alors } AV + b \sim N(Am + b,\ A \Sigma A^T) \quad \operatorname{rg} A = p \text{ (nbre de ligne)}}$

$\operatorname{rg} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = 1$

$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ vecteur gaussien car X, Y, Z ind

$N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$

$\Downarrow$

$$U = A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + b \sim N(Am + b,\ A \Sigma A^T)$$

$Am + b = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + 0 = 0$

$A \Sigma A^T = \underbrace{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} I_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{3}$

donc $\boxed{U \sim N(0, 3)}$

2) Montrer que $V = X - Y$ et $U = X + Y + Z$ sont des variables aléatoires indépendantes

Remarque

$$\begin{pmatrix} U \\ V \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$\operatorname{rg} A = 2$ donc $\begin{pmatrix} U \\ V \end{pmatrix}$ est un vecteur gaussien (car transformation affine d'un vecteur gaussien)

donc $\boxed{U \text{ et } V \text{ ind} \iff \operatorname{cov}(U,V) = 0}$

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \sim N_2\left(\underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{Am+b}, \; A \Sigma A^T \right)$$

$\operatorname{rg}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 2$

Si $V = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ est un vecteur gaussien alors X, Y et Z sont des variables gaussiennes

La réciproque est fausse (voir TDG)

$$A \Sigma A^T = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

donc $\begin{pmatrix} U \\ V \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \right)$

$\left.\begin{array}{l} \begin{pmatrix} U \\ V \end{pmatrix} \text{ vecteur gaussien} \\ \operatorname{cov}(U,V) = 0 \end{array}\right\} \Rightarrow U \text{ et } V \text{ indépendants}$

Ex03

$X \sim N(0,1)$
$Y \sim N(0,1)$
X et Y indépendants
chercher $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ et $n = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ tels que

$V = M \begin{pmatrix} X \\ Y \end{pmatrix} + n \sim N(m, \Sigma)$ m et Σ donnés

par exemple
$m = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

si $\mathrm{rg}\, M = 2$, on a $V \sim N\left(M \begin{pmatrix} 0 \\ 0 \end{pmatrix} + n, \; M \Sigma_{\binom{X}{Y}} M^T \right)$

$V \sim N(n, MM^T)$ car $\Sigma_{\binom{X}{Y}} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

si on veut $V \sim N(m, \Sigma)$ il suffit de choisir n et M

tels que $n = m$ $\left(= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ pour l'exemple} \right)$

$MM^T = \Sigma$ $\left(\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \text{ pour l'exemple} \right)$

$$\boxed{\Sigma = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^T}$$

matrice symétrique
déf > 0 donc diagonalisable

P, $\binom{\lambda_1}{\lambda_2}$ sont fournis par la routine eig.m (sous matlab)

$$M = P \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} P^T \leftarrow$$

$$MM^T = P \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} \underbrace{P^T P}_{I} \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} P^T$$

$$= P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^T = \Sigma$$

$\boxed{\text{Exo 6}}$ $X_j \sim P(\theta)$ $\theta = 1$

1) Loi de $S_n = \sum_{j=1}^{n} X_j$?

2) 2) $T_n = \frac{S_n - n}{\sqrt{n}}$

Théorème de la limite centrale

En considérant $T_n < 0$, montrer que $\lim_{n \to +\infty} e^{-n} \sum_{k=0}^{n-1} \frac{n^k}{k!} = \frac{1}{2}$

$$\frac{S_n - ? \; E[S_n]}{? \; \sqrt{\operatorname{var} S_n}} \xrightarrow[n \to +\infty]{} N(0,1)$$

1) <u>Fonction caractéristique</u>

$$\phi_{S_n}(t) = E[e^{itS_n}] = E\left[e^{it\sum_{j=1}^{n} X_j}\right]$$

$$= E\left[\prod_{j=1}^{n} e^{itX_j}\right]$$

$$\underset{\substack{X_1,\dots,X_n \text{ ind} \\ X_j \sim P(\lambda)}}{=} \prod_{j=1}^{n} \underbrace{E[e^{itX_j}]}_{\substack{\phi_{X_j}(t) = \exp[e^{it}-1] \\ (\text{Table})}}$$

$$= \exp[n(e^{it}-1)]$$

qui est la fonction caractéristique d'une loi de Poisson $P(n)$

donc $\boxed{S_n \sim P(n)}$

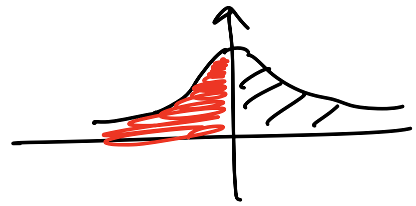
$$S_n = \sum_{j=1}^{n} X_j$$

$$E[S_n] = \sum_{j=1}^{n} \underbrace{E(X_j)}_{1} = n$$

$$Var[S_n] = Var\left(\sum_{j=1}^{n} X_j\right) \underset{X_1,\dots,X_n \text{ ind}}{=} \sum_{j=1}^{n} \underbrace{Var(X_j)}_{1} = n$$

$$T_n = \frac{S_n - n}{\sqrt{n}} \xrightarrow[n \to +\infty]{} N(0,1)$$

$$P[T_n < 0] \xrightarrow[n \to +\infty]{} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du = \frac{1}{2}$$



$$P[T_n < 0] = P[S_n < n]$$

$$= P[S_n = 0, \text{ ou } 1 \text{ ou } \ldots \ldots \text{ ou } n-1]$$

$$= \sum_{k=0}^{n-1} \underbrace{P[S_n = k]}_{S_n \sim P(n)}$$

$$\boxed{\begin{array}{c} P[X=k] = \dfrac{\lambda^k}{k!} e^{-\lambda} \\ k \in \mathbb{N} \\ \text{Loi de Poisson } P(\lambda) \end{array}}$$

donc $P[S_n = k] = \dfrac{n^k}{k!} e^{-n}$

$$\boxed{\sum_{k=0}^{n-1} \left(\frac{n^k}{k!} e^{-n} \right) \xrightarrow[n \to +\infty]{} \frac{1}{2}}$$

THE END!!