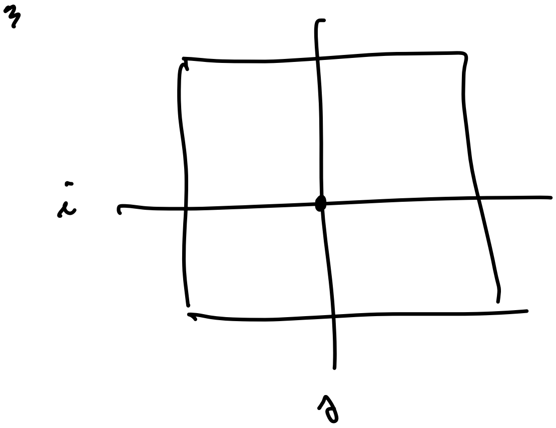
Cours 11/09/2023



$$\overset{\in\{0,\dots,255\}}{Z_{ij}} = \underset{\text{vérité terrain}}{V_{ij}} + \underbrace{e_{ij}}_{\text{"bruit"}}$$

X

$(\Omega, \mathcal{C}, P)$ expérience aléatoire

jet de dé

- Ω = ensemble des résultats d'expérience
  - jet de dé $\Omega = \{1, \dots, 6\}$
- $\mathcal{C}$ = ensemble des événements
  - $\mathcal{C} = \{ \{1\}, \dots, \{6\}, \{1,2\}, \dots, \{1,2,\dots,6\}, \emptyset \}$ ← événement impossible
- $P : \mathcal{C} \to [0,1]$

$$\boxed{P\left(\begin{array}{c}k \text{ succès sur } n \text{ expériences} \\ \text{de type "succès" ou "échec"}\end{array}\right) = \underbrace{\binom{n}{k}}_{\frac{n!}{k!(n-k)!}} P_S^{k} (1-P_S)^{n-k} \quad k \in \{0, \dots, n\}}$$

P_S = probabilité du succès sur 1 expérience

EQUIPROBABILITÉ :

- Ω fini
- les événements élémentaires $\{a\}$, $a \in \Omega$ ont la même importance

$$P(A) = \frac{\text{Card } A}{\text{Card } \Omega} = \frac{\text{nombre de cas favorables}}{\text{nombre de cas possibles}}$$

Card Ω   [illegible]

Exemple

10 jets de dé $n = 10$ succès = "1"

$P(\underbrace{3 \text{ fois } 1}_{3 \text{ succès } k=3}) = \frac{10!}{3!\,7!} p_S^3 (1-p_S)^7$ échec = $\{2, \dots, 6\}$

$p_S = 1/6$

Probabilités conditionnelles

$$P(A|B) = \boxed{\frac{P(A \cap B)}{P(B)}}$$

si équiprobabilité $= \dfrac{\frac{\text{Card}(A\cap B)}{\cancel{\text{card}\,\Omega}}}{\frac{\text{Card}\,B}{\cancel{\text{card}\,\Omega}}} = \dfrac{\text{Card}(A\cap B)}{\text{Card}\,B}$

Corollaire $\boxed{P(A \cap B) = P(A|B)\,P(B)}$

A et B ind $\iff$ $\boxed{P(A\cap B) = P(A) \times P(B)}$

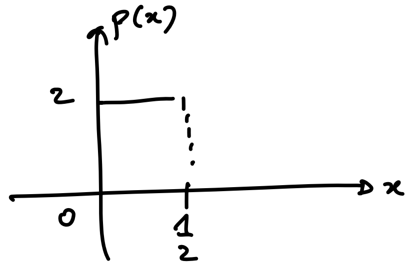
$P(A) \neq 0$, $P(B) \neq 0$ $\iff$ " $P(A|B) = P(A)$ "

Formule de BAYES $\boxed{P(A|B) = \dfrac{P(B|A)\,P(A)}{P(B)}}$

$\underbrace{P(A|B) = \frac{P(A\cap B)}{P(B)}}$ $\Rightarrow$ $P(A\cap B) = P(A|B)\,P(B)$

$\underbrace{P(B|A) = \frac{P(B\cap A)}{P(A)}}$ $\Rightarrow$ $P(B\cap A) = P(B|A)\,P(A)$

$\Rightarrow$ $\boxed{\text{CQFD}}$

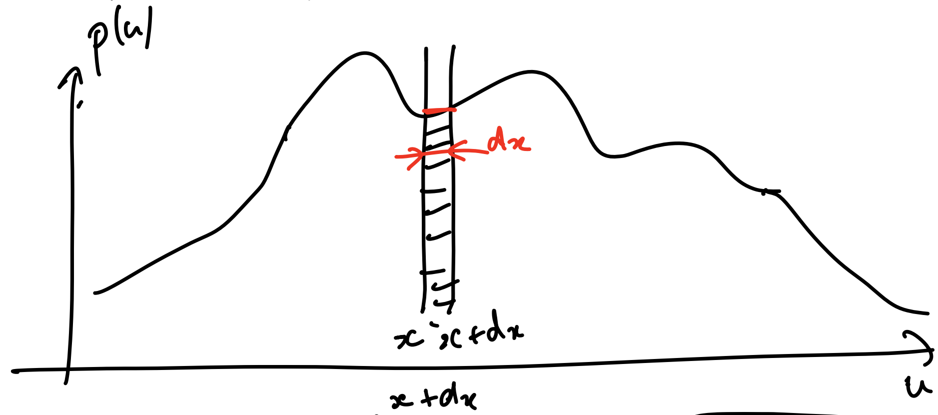


$$p(x) = \begin{cases} 2 & x \in]0, \frac{1}{2}[\\ 0 & \text{sinon} \end{cases}$$

donc on peut avoir $p(x) > 1$

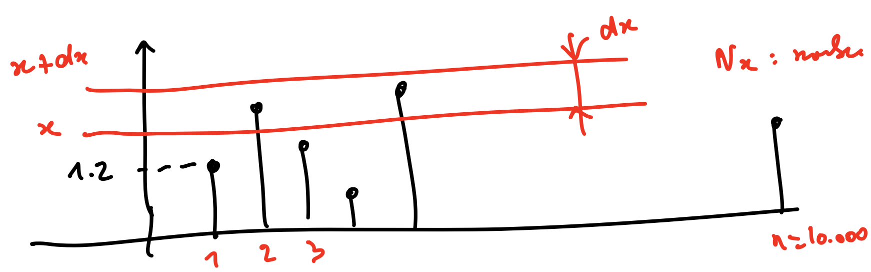
$$P[X \in \Delta] = \int_\Delta p(u)\, du$$

$$\boxed{P[X \in]x, x+dx[]} = \int_x^{x+dx} p(u)\, du$$

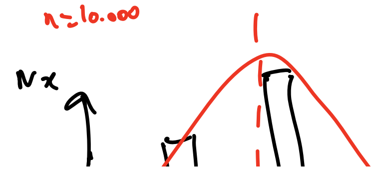


$$\underset{dx \text{ petit}}{\simeq} \int_x^{x+dx} p(x)\, du = \boxed{p(x)\, dx}$$

$$\boxed{p(x) \simeq \frac{P[X \in]x, x+dx[]}{dx}}$$

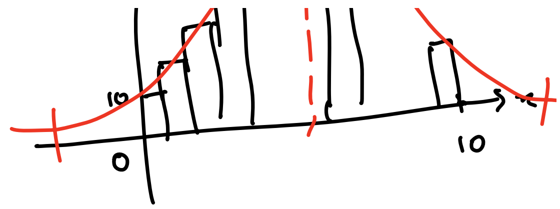


N_x : nombre de points $\in]x, x+dx[$



$$P[X \in]x, x+dx[) \simeq \frac{N_x}{n}$$

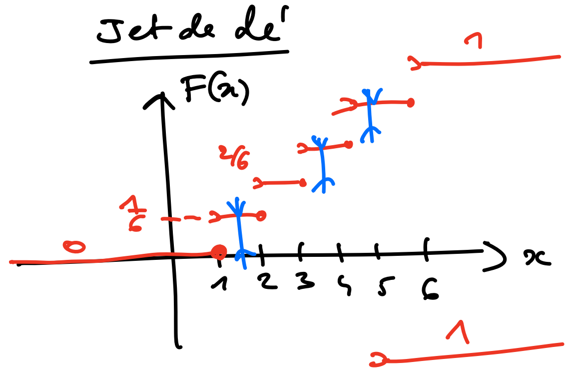
$$\boxed{p(x) \simeq \frac{N_x}{n\,dx}}$$

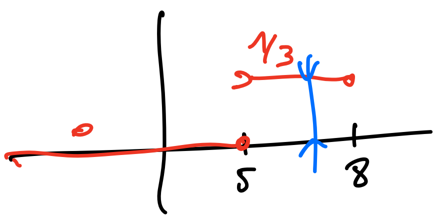


Fonction de répartition

$$F(x) = P[X \le x]$$

Ex 1 Jet de dé



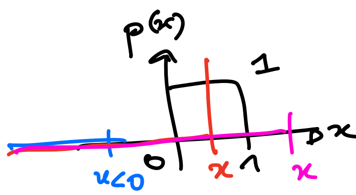


X va à valeurs dans $\{5, 8\}$

$P[X=5] = \frac{1}{3}$

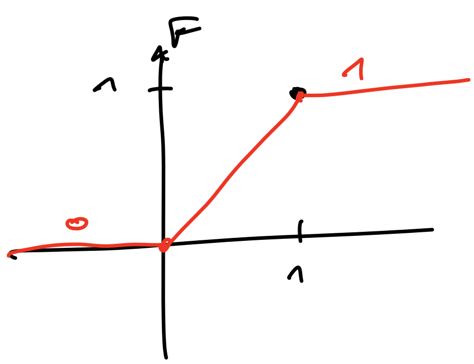
$P[X=8] = \frac{2}{3}$

Ex 2 X uniforme sur $]0,1[$



$$\boxed{F(x)} = P(X < x) \overset{\Delta}{=} P[X \in]-\infty, x[]$$

$$= \boxed{\int_{-\infty}^{x} p(u)\,du} \quad \text{donc} \quad \boxed{p(x) = F'(x)}$$



$$= \begin{cases} 0 & x \le 0 \\ \int_0^x 1\,du = x & x \in]0,1[\\ 1 & x \ge 1 \end{cases}$$

Espérance Mathématique

① Jet de dé $E[X] = (1\times\frac{1}{6})+(2\times\frac{1}{6})+\dots+(6\times\frac{1}{6})$

$$= \frac{1}{6}(\underbrace{1+2+\dots+6}_{\frac{6\times 7}{2}}) = \frac{7}{2} = \boxed{3.5}$$

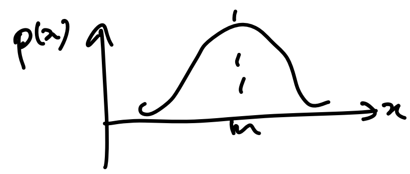
Rq : on verra que si $x_1^*, \dots, x_n^*$ sont n réalisations de X, sous certaines hypothèses

Loi des grands nombres

$$\underbrace{\frac{x_1^*+\dots+x_n^*}{n}}_{\text{moyenne arythmétique}} \xrightarrow[n\to\infty]{} E[X]$$

② Loi Normale

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] \quad x\in\mathbb{R}$$



$$E(X) = \int_{\mathbb{R}} u\,\boxed{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(u-m)^2}{2\sigma^2}\right)du}$$

$$= \int_{\mathbb{R}} (\sigma v+m)\,\boxed{\frac{1}{\sqrt{2\pi\cancel{\sigma}^2}}\exp\left(-\frac{v^2}{2}\right)\cancel{\sigma}\,dv}$$

$v = \frac{u-m}{\sigma}$

$u = \sigma v + m$

$$= \sigma\underbrace{\int v\,\frac{1}{\sqrt{2\pi}}e^{-v^2/2}} + m\underbrace{\int \frac{1}{\sqrt{2\pi}}e^{-v^2/2}dv}_{1}$$

0

intégrale d'une fonction impaire

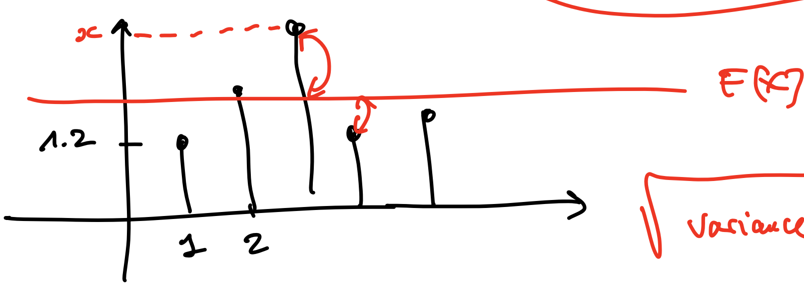
$$\boxed{E[X] = m}$$

moments centrés

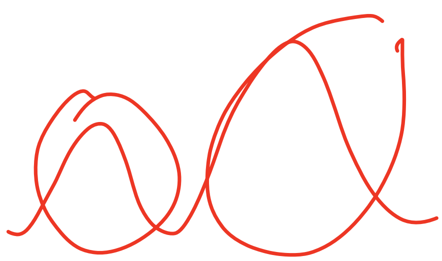
n = 1 $\quad E[X - \overbrace{E[X]}^{\text{constante}}]$

$$E(X) - \underbrace{E[E(X)]}_{E(X)} = \boxed{0}$$

n = 2 $\quad \underbrace{E[(X - E[X])^2]}$ Variance



$\sqrt{\text{variance}}$ = écart-type



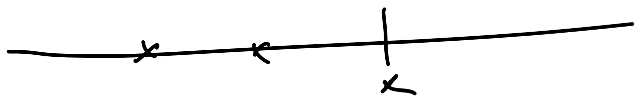
Rq $\quad \boxed{E[(X - E[X])^2]} = E[X^2] - 2E[X] \times E[X] + E^2[X]$

$$X^2 - 2XE[X] + E^2[X]$$

$$= \boxed{E(X^2) - E^2(X)}$$ Calcul

$\boxed{\text{Cours du } 12/03/2023}$

$$X = \mathrm{Max}\{X_1, X_2\}$$



$$P(X < x) = P[X_1 < x,\ X_2 < x]$$

$$\overset{X_1 \text{ et } X_2 \text{ iid}}{=} P[X_1 < x] \times P[X_2 < x]$$

<u>Variance de la loi Normale</u>

$$\boxed{\begin{array}{l} X \sim \mathcal{N}(\underbrace{m}_{\text{moyenne } E(X)}, \sigma^2) \\ \mathrm{Var}(X)\ ? \end{array}}$$

$$\mathrm{Var}\,X = E(X^2) - E^2[X]$$

On a vu hier $E(X) = m$

$$E(X^2) = \int x^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] dx$$

$u = \frac{x-m}{\sigma}$

$x = \sigma u + m$

On centre et on réduit

$$\overset{u = \frac{x-m}{\sigma}}{=} \int (\sigma u + m)^2 \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

$$= \sigma^2 \underbrace{\int u^2 \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du}_{\text{se calcule par intégration par parties}} + 2m\sigma \underbrace{\int u \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du}_{=0}$$

$$+ m^2 \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du}_{1}$$

<u>IPP</u>

$$\begin{cases} v' = u e^{-u^2/2} \\ w = \frac{u}{\sqrt{}} \end{cases} \quad \ldots \quad \Rightarrow \boxed{\int \frac{u^2}{\sqrt{2\pi}} e^{-u^2/2}\, du = 1}$$

$\sqrt{2\pi}$

$$E(X^2) = \sigma^2 + m^2$$

$$\mathrm{Var}\,X = E(X^2) - E^2(X) = \boxed{\sigma^2}$$

CQFD

$$\boxed{\text{Loi de } X = X_1 + X_2 \text{, } X_1 \text{ et } X_2 \text{ sont ind}}$$

$$\begin{aligned}
\phi_X(u) &= E\left[e^{iuX}\right] \\
&= E\left[e^{iuX_1}\, e^{iuX_2}\right] \\
&\underset{X_1 \text{ et } X_2 \text{ ind}}{=} \underbrace{E\left[e^{iuX_1}\right]}_{\phi_{X_1}(u)}\, E\left[e^{iuX_2}\right] \\
& \qquad \phi_{X_1}(u)\, \phi_{X_2}(u)
\end{aligned}$$

$$\left[\begin{array}{l} X_1 \sim N(m_1, \sigma_1^2) \\ X_2 \sim N(m_2, \sigma_2^2) \end{array}\right. \qquad \begin{array}{l} \phi_{X_1}(u) = e^{i m_1 u - \sigma_1^2 \frac{u^2}{2}} \\ \phi_{X_2}(u) = e^{i m_2 u - \sigma_2^2 \frac{u^2}{2}} \end{array}$$

$$\phi_X(u) = \phi_{X_1}(u)\, \phi_{X_2}(u)$$

$$= e^{i(m_1+m_2)u - (\sigma_1^2+\sigma_2^2)\frac{u^2}{2}}$$

$$\Rightarrow \boxed{X \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)}$$

Changement de variable

$$X \sim \mathcal{P}(\lambda) \qquad P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad k \in \mathbb{N}$$

loi de $Y = (X-2)^2$

X	Y
0	4
1	1
2	0
3	1
4	4
5	9
6	16
7	25

⋮

- Y est à valeurs dans $\{k^2,\ k \in \mathbb{N}\}$
- $P[Y = k^2] = P((X-2)^2 = k^2)$

$$= P(X = 2+k \text{ ou } X = 2-k]$$

1er cas $\quad 2+k = 2-k \iff \boxed{k=0}$

$$P[Y=0] = P[X=2] = \frac{\lambda^2}{2} e^{-\lambda}$$

2ème cas $\quad \boxed{k \neq 0}$

$$P[Y=k^2] = \underbrace{P[X=2+k]}_{\frac{\lambda^{2+k}}{(2+k)!} e^{-\lambda}} + P(X=2-k]$$

$2-k \geq 0 \iff k \leq 2$

si $k>2$ alors $P[X=2-k] = 0$

si $k=1$ $\quad P(X=2-k) = P[X=1]$

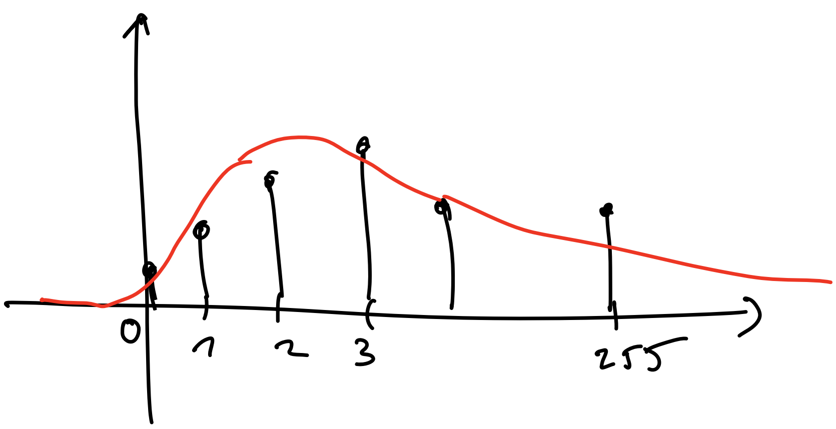
$$= \lambda e^{-\lambda}$$

si $\underline{k=2}$ $\quad P[X=2-k] = P[X=0]$

$$= e^{-\lambda}$$

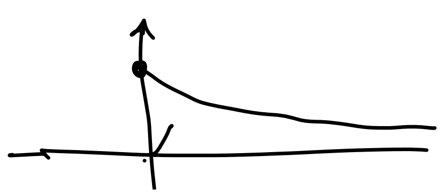
<u>Conclusion</u>

$$P[Y=k^2] = \begin{cases} \underline{k=0} & \dfrac{\lambda^2}{2} e^{-\lambda} \\ \underline{k=1} & \lambda e^{-\lambda} + \dfrac{\lambda^3}{6} e^{-\lambda} \\ k=2 & e^{-\lambda} + \dfrac{\lambda^4}{24} e^{-\lambda} \\ k \geq 3,\ k \in \mathbb{N} & \dfrac{\lambda^{2+k}}{(2+k)!} e^{-\lambda} \end{cases}$$



<u>changement de variable</u>

$X \sim \mathcal{E}(1)$ $\qquad p(x) = \begin{cases} e^{-x} & x>0 \\ 0 & x \leq 0 \end{cases}$



Loi de $Y = \frac{1}{X}$?

- Il est clair que Y est une va continue à valeurs dans $]0, +\infty[$

- <u>Densité de Y notée $\pi(y)$</u>

<u>Recette de cuisine</u>

$$\boxed{Y = \frac{1}{X} \Leftrightarrow X = \frac{1}{Y}}$$

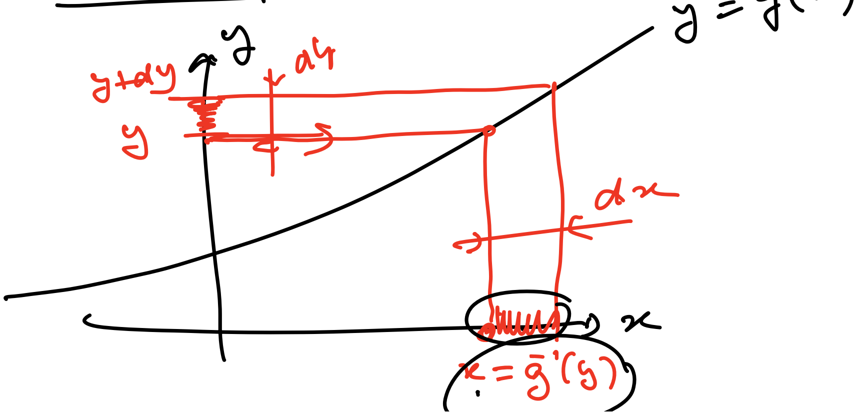
$$\left(-\frac{1}{y^2}\right)$$

$$\pi(y) = e^{-\frac{1}{y}} \underbrace{\left|\frac{dx}{dy}\right|}_{\text{Jacobien de la transformation}}$$

on fait $x(y)$ dans $p(x)$

$$\boxed{\pi(y) = \begin{cases} \frac{1}{y^2} e^{-\frac{1}{y}} & y > 0 \\ 0 & \text{sinon} \end{cases}}$$

<u>Idée de preuve</u>



$$\pi(y)\,dy \simeq P[Y \in]y, y+dy[]$$

dy petit

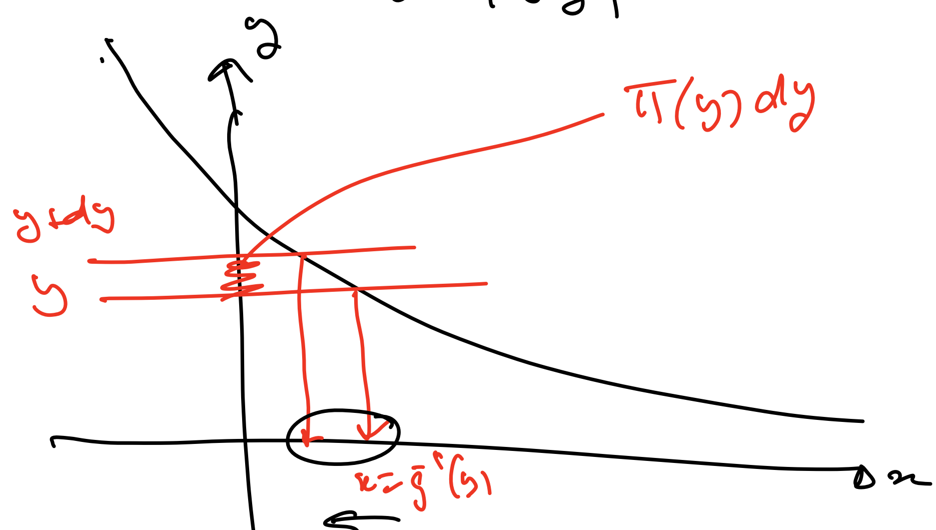
$$\pi(y)\,dy = p_X[g^{-1}(y)]\,dx$$

$$\boxed{\pi(y) = p_X[g^{-1}(y)]\,\frac{dx}{dy}}$$

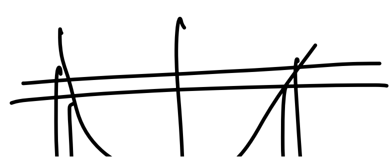
$$P[X \in]x, x+dx[] = p_X(x)\,dx$$

Rq ici $\frac{dx}{dy} > 0$ car la fonction est croissante

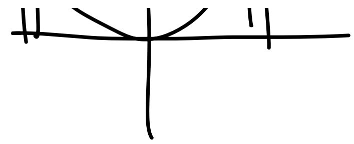
$$\frac{dx}{dy} = \left|\frac{dx}{dy}\right|$$



$dx < 0 \Rightarrow \left|\frac{dx}{dy}\right| = -\frac{dx}{dy}$



PREUVE



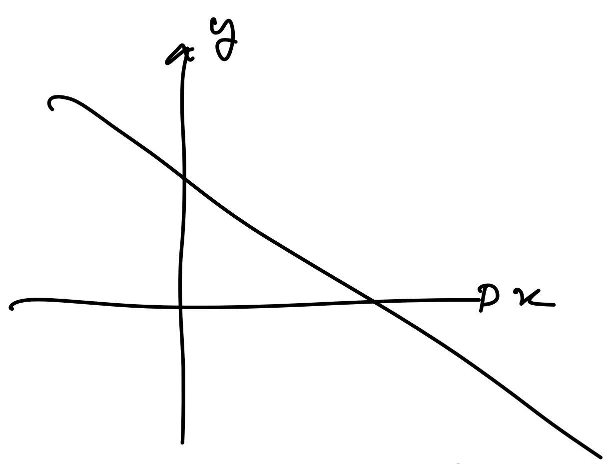
$$P[Y \in \Delta] = \int_\Delta \pi(y)\,dy \qquad = \forall \Delta$$

$$P(Y \in \Delta) = P[g(X) \in \Delta]$$
$$= P(X \in g^{-1}(\Delta)]$$
$$= \int_{g^{-1}(\Delta)} p_X(x)\,dx$$
$$= \int_\Delta p_X[g^{-1}(y)] \left|\frac{dx}{dy}\right| dy$$

$x = g^{-1}(y)$
$\in g^{-1}(\Delta)$

donc $\boxed{\pi(y) = p_X[g^{-1}(y)] \left|\frac{dx}{dy}\right|}$

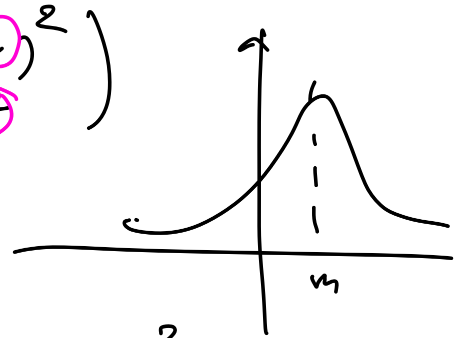
$X \sim N(m, \sigma^2)$
loi de $Y = aX + b$
$a \neq 0$



C'est une transformation bijective de $\mathbb{R}$ dans $\mathbb{R}$

$$Y = aX + b \iff X = \frac{Y - b}{a}$$

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$



Densité de Y

$$\Pi(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\left(\frac{y-b}{a}-m\right)^2}{2\sigma^2}\right] \left|\frac{1}{a}\right|$$

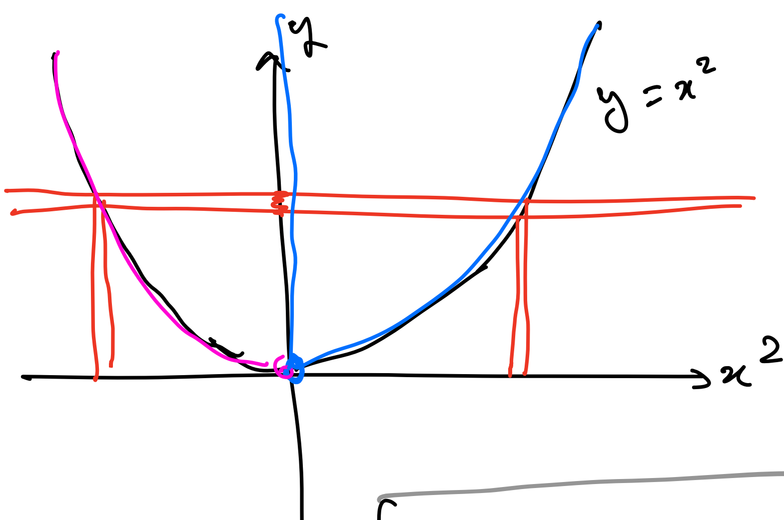
$$= \frac{1}{\sqrt{2\pi a^2\sigma^2}} \exp\left\{-\frac{1}{2a^2\sigma^2}\left[y-\underbrace{(b+am)}_{\mu}\right]^2\right\}$$

$$\boxed{Y \sim N(am+b,\ a^2\sigma^2)}$$

Rq $y = ax+b \Rightarrow E(Y) = a\,\underbrace{E(X)}_{m} + b$

$\mathrm{Var}\,Y = a^2\,\mathrm{Var}\,X$

$= a^2\sigma^2$



Exemple $X \sim N(0,1)$ $\left(p(x) = \frac{1}{\quad} e^{-x^2/2} \right.$

loi de $Y = X^2$ $\sqrt{2\pi}$

Bijection 1

$\mathbb{R}^+ \longrightarrow \mathbb{R}^+$

$x \longmapsto y = x^2 \iff x = \sqrt{y}$

$$\pi_1(y) = \frac{1}{\sqrt{2\pi}} e^{-y/2} \left| \frac{dx}{dy} \right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2\sqrt{y}}$$

Bijection 2

$\mathbb{R}^- \longrightarrow \mathbb{R}^+$

$x \longmapsto y = x^2 \iff x = -\sqrt{y}$

$$\pi_2(y) = \frac{1}{\sqrt{2\pi}} e^{-y/2} \left| -\frac{1}{2\sqrt{y}} \right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2\sqrt{y}}$$

Conclusion

$$\boxed{\pi(y) = \pi_1(y) + \pi_2(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}} e^{-y/2} & y \geq 0 \\ 0 & \text{sinon} \end{cases}}$$

$Y \sim \chi^2_1$ loi du chi-deux à 1 degré de liberté.

$\boxed{\text{Cours du 19/09/2023}}$

Exemple

$$(X,Y) \to \begin{cases} (0,0) & \frac{1}{2} \\ (0,1) & \frac{1}{6} \\ (1,0) & \frac{1}{6} \\ (1,1) & \frac{1}{6} \end{cases}$$

y \ x	0	1
0	1/2	1/6
1	1/6	1/6

Loi de X

$$X \to \begin{cases} 0 & 2/3 \\ 1 & 1/3 \end{cases}$$

$$P(X=0) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$P(X=1) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

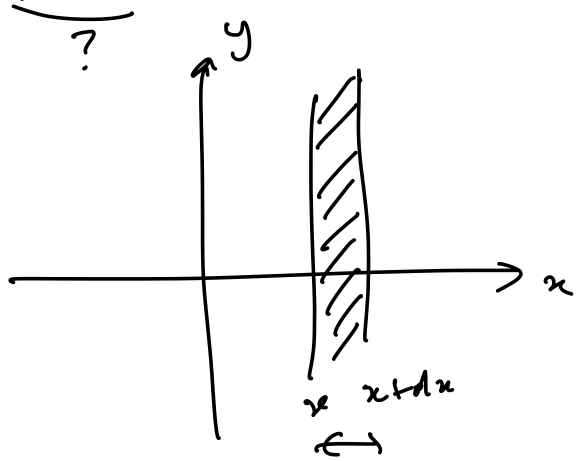
$$Y \to \begin{cases} 0 \\ 1 \end{cases}$$

$\boxed{\begin{array}{l} (X,Y) \text{ de densité } p(x,y) \\ \text{Quelle est la densité de } X \text{ ?} \end{array}}$

$$P[\underbrace{X \in]x, x+dx[}_{(X,Y) \in \Delta}] \simeq \underbrace{p(x,\cdot)}_{?}\, dx$$

$$\| $$

$$\int_x^{x+dx} \int_{\mathbb{R}} p(u,v)\, du\, dv$$



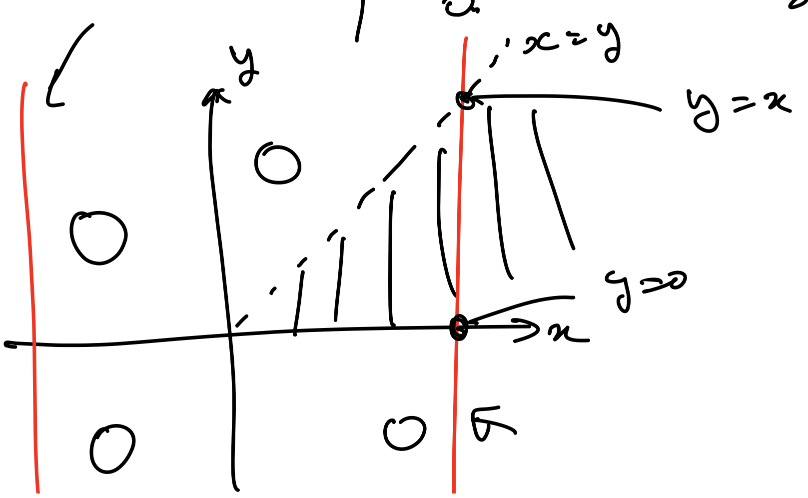
dx "petit" $\qquad p(u,v) \approx p(x,v) \quad \forall (u,v)$

$$\approx \int_x^{x+dx} \int_{\mathbb{R}} p(x,v)\, du\, dv$$

$$= \int_x^{x+dx} \underbrace{\left[\int_{\mathbb{R}} p(x,v)\, dv \right]}_{\text{ind de } u} du$$

$$= \underbrace{\int_{\mathbb{R}} p(x,v)\, dv}_{p(x,\cdot)} \; \underbrace{\int_x^{x+dx} du}_{dx}$$

$$p(x,y) = \begin{cases} \theta^2 e^{-\theta x} & x > y > 0 \\ 0 & \text{sinon} \end{cases}$$



Loi de X il est clair que X est à valeurs dans $]0,+\infty[$

$$p(x,\cdot) = \int_{\mathbb{R}} p(x,y)\,dy$$

$$= \begin{cases} 0 & x \leq 0 \\ \underbrace{\int_0^x \theta^2 e^{-\theta x}\,dy}_{\theta^2 x e^{-\theta x}} & x > 0 \end{cases}$$

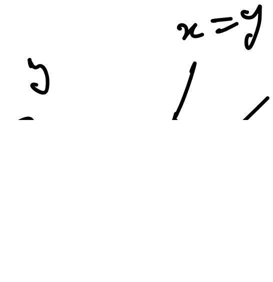
$$\boxed{p(x,\cdot) = \theta^2 x e^{-\theta x} \underbrace{I_{]0,+\infty[}(x)}}$$

$= 1$ $x \in]0,+\infty[$

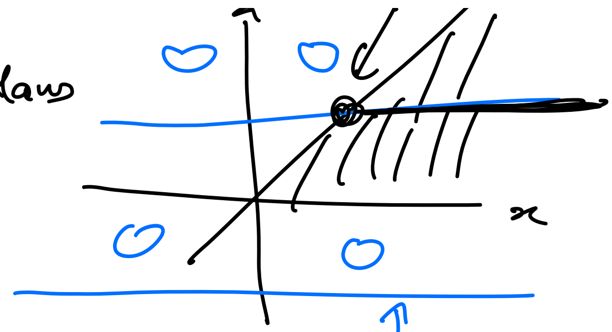
$= 0$ sinon

$$\boxed{p(x,\cdot) = \begin{cases} \theta^2 x e^{-\theta x} & x > 0 \\ 0 & \text{sinon} \end{cases}}$$

donc $\boxed{X \sim \Gamma(\theta, 2)}$



Loi de Y : Y est à valeurs dans $]0,+\infty[$



La densité de Y est

$$p(\cdot,y) = \int_{\mathbb{R}} p(x,y)\,dx$$

$$= \begin{cases} 0 & y \leq 0 \\ \displaystyle\int_y^{+\infty} \theta^2 e^{-\theta x}\,dx & y > 0 \end{cases}$$

$$\left[-\theta e^{-\theta x}\right]_y^{+\infty} = \theta e^{-\theta y}$$

$$\boxed{p(\cdot,y) = \begin{cases} \theta e^{-\theta y} & y>0 \\ 0 & \text{sinon} \end{cases}}$$

donc $\boxed{Y \sim G(\theta,1)}$

Espérance mathématique

$\underline{1D}$ $E[X]$ Moyenne

$Var(X) = E[X^2] - E^2[X]$ Variance

$E[e^{itX}] = \varphi(t)$ Fonction caractéristique

Exemple d'utilisation de la fonction caractéristique

$X \sim N(m_1, \sigma_1^2) \longrightarrow \varphi_X(t) = e^{im_1 t - \sigma_1^2 \frac{t^2}{2}}$

$Y \sim N(m_2, \sigma_2^2) \longrightarrow \varphi_Y(t) = e^{im_2 t - \sigma_2^2 \frac{t^2}{2}}$

loi de $Z = X+Y$

$$E[e^{itZ}] = E[e^{it(X+Y)}] = E[e^{itX} e^{itY}]$$

X et Y ind $\Rightarrow e^{itX}$ et e^{itY} ind

$$\Rightarrow E[e^{itX} e^{itY}] = E[e^{itX}] \times E[e^{itY}]$$

$$\boxed{\varphi_Z(t) = \varphi_X(t)\,\varphi_Y(t)}$$

donc $\varphi_Z(t) = e^{i\overbrace{(m_1+m_2)}^{m}t - \frac{1}{2}\overbrace{(\sigma_1^2+\sigma_2^2)}^{\sigma^2}t^2}$

$$\boxed{Z \sim N(m_1+m_2,\ \sigma_1^2+\sigma_2^2)}$$

à 2D

$E[XY]$?

covariance $\quad E[XY] - E(X)\,E[Y]$

Si $X = Y$, on retrouve la définition de la variance

à 2D

$E(X) \rightarrow \sum x_i\, P(X = x_i)$

$E(X) \rightarrow \int x\, p(x)\,dx$

à 2D

$E(XY) \rightarrow \sum_{i,j} x_i y_j\, P(X = x_i, Y = y_j)$

$E(XY) \rightarrow \iint xy\, p(x,y)\,dx\,dy$

Définitions cohérentes

$$(X,Y) \longrightarrow p(x,y) \quad = \int x \Big[\underbrace{\int p(x,y)\,dy}_{p(x,\cdot)} \Big] dx$$

$$E(X) = E[\alpha(X,Y)] = \iint x\, p(x,y)\,dx\,dy$$

$$(X,Y) \longrightarrow p(x,y) \longrightarrow p(x,\cdot)$$

$$E(X) = \int x\, p(x,\cdot)\,dx$$

Indépendance

si X et Y ind $\quad E(\alpha(X)\beta(Y)) = E[\alpha(X)] \times E[\beta(Y)]$

$$E(\alpha(X)\beta(Y)) = \iint \alpha(x)\beta(y)\, p(x,y)\,dx\,dy$$

$$= \iint \alpha(x)\,\beta(y)\,p(x,\cdot)\,p(\cdot,y)\,dx\,dy$$

X et Y ind

$$= \underbrace{\left(\int \alpha(x)\,p(x,\cdot)\,dx\right)}_{E(\alpha(X))}\underbrace{\left(\int \beta(y)\,p(\cdot,y)\,dy\right)}_{E[\beta(Y)]}$$

Cas particulier

$$\boxed{X \text{ et } Y \text{ ind } \underset{\not\Leftarrow}{\Rightarrow} \underbrace{Cov(X,Y)}_{E(XY)-E(X)E(Y)} = 0}$$

Matrice de covariance

$$M = \begin{pmatrix} Var\, X & cov(X,Y) \\ cov(X,Y) & Var\, Y \end{pmatrix} \qquad V = \begin{bmatrix} X - E(X) \\ Y - E(Y) \end{bmatrix}$$

$$M = E[V\, V^T]$$

$$V\, V^T = \begin{bmatrix} X - E(X) \\ Y - E(Y) \end{bmatrix} \begin{bmatrix} X - E(X) & Y - E(Y) \end{bmatrix}$$

$$= \begin{bmatrix} (X - E(X))^2 & (X - E(X))(Y - E(Y)) \\ (Y - E(Y))(X - E(X)) & (Y - E(Y))^2 \end{bmatrix}$$

donc $\boxed{M = E[V\, V^T]}$

$$\mathrm{cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$\mathrm{cov}(10X, 10Y) = 100\ \mathrm{cov}(X,Y)$$

$$\boxed{r_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}X}\ \sqrt{\mathrm{Var}Y}}}$$

coefficient de corrélation

$$\boxed{r_{10X,10Y} = r_{X,Y}}$$

intéressant

En plus $r_{X,Y} \in]-1, +1)$

Rq : $\langle X, Y \rangle = E(XY)$ définit un produit scalaire

$\mathrm{cov}(X,Y) = \langle X - E[X], Y - E(Y) \rangle$

CAUCHY SHWARTZ

$$|\langle X, Y \rangle|^2 \leq \|X\|^2 \|Y\|^2$$

égalité si $aX + bY + c = 0$

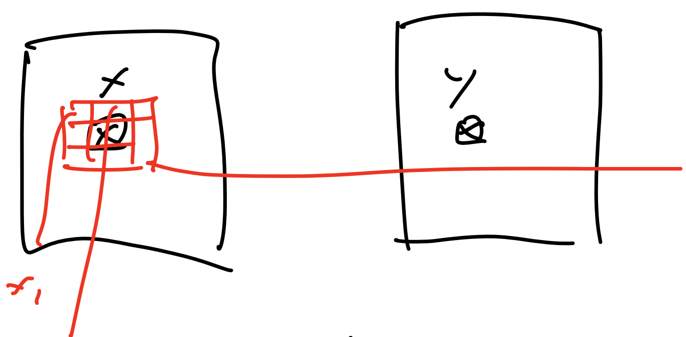
$\langle X, X \rangle$

Var Y

$$\underbrace{\left|\langle X-E(Y),\ Y-E(Y)\rangle\right|^2}_{\mathrm{cov}^2(X,Y)} \le \underbrace{\|X-E(X)\|^2}_{\substack{\langle X-E(X),\, X-E(X)\rangle \\ \mathrm{Var}\,X}}\ \|Y-E(Y)\|^2$$

$$\boxed{r_{X,Y}^2 = \frac{\mathrm{cov}^2(X,Y)}{\mathrm{Var}\,X\ \mathrm{Var}\,Y} \le 1}$$

$$\boxed{\begin{gathered}\mathrm{Cov}(X,Y) = E[XY]-E[X]E[Y] \\ r_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}X}\sqrt{\mathrm{Var}Y}}\end{gathered}}$$



$$E(X) \approx \frac{x_1+\dots+x_9}{9}$$

x_2 si $|r_{X,Y}| >$ seuil alors pas de changement.

$<$ seuil alors changement

$$r_{XY} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{Var}X}\sqrt{\mathrm{Var}Y}} = \frac{E(XY)-E(X)E(Y)}{\sqrt{E(X^2)-E^2(X)}\sqrt{E(Y^2)-E^2(Y)}}$$

$E(X)$

$E(Y)$

$E(X^2)$

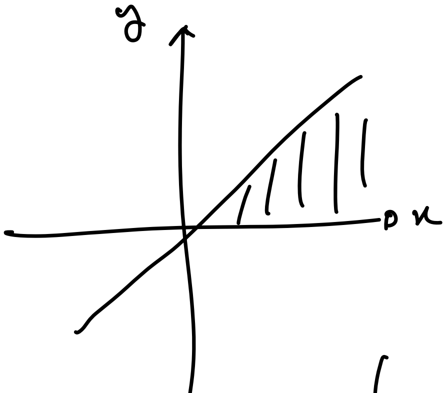
$E(Y^2)$

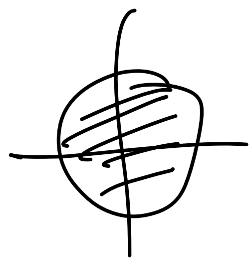
$E(XY) \approx \frac{1}{9}(x_1y_1 + \dots + x_9y_9)$

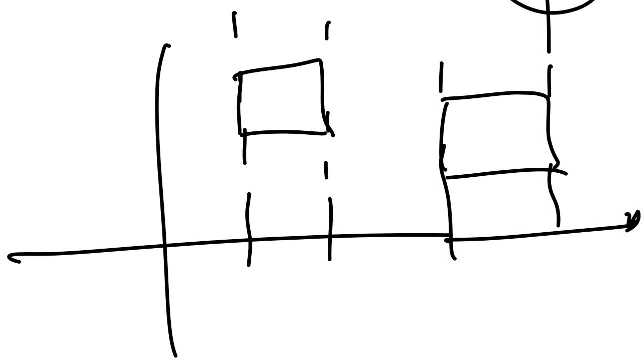
Cours du 26/09/2023

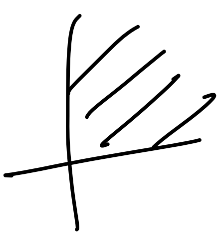
Cas continu

X et Y indépendantes $\iff p(x,y) = p(x,\cdot) \times p(\cdot,y)$ $\forall x\ \forall y$









Espérances conditionnelles -

$$E[\alpha(X,Y)] = E_X\left[\, E_Y[\alpha(X,Y) \mid X] \,\right]$$

Preuve (cas continu)

$$E\left[\alpha(X,Y)\right] = \iint \alpha(u,v)\, p(u,v)\, du\, dv$$

$$p(v|u) = \frac{p(v,u)}{p(u,\cdot)}$$

$$= \iint \alpha(u,v)\, p(v|u)\, p(u,\cdot)\, du\, dv$$

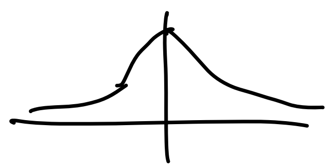
$$= \int \left[\int \alpha(u,v)\, p(v|u)\, p(u,\cdot)\, dv \right] du$$

$$= \int p(u,\cdot) \underbrace{\left[\int \alpha(u,v)\, p(v|u)\, dv \right]}_{E\left[\alpha(X,Y)\,|\,X\right]} du$$

$$= E_X\left[E_{Y|X}\left[\alpha(X,Y)\,|\,X\right] \right]$$

Exemple 1

$X \sim N(0,1) \qquad p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$



$$Y \begin{cases} \to 1 & p \\ \to -1 & q = 1-p \end{cases}$$

loi de $Z = XY$?

Fonction caractéristique de Z

$$\varphi_Z(u) = E[e^{iZu}]$$
$$= E[e^{iXYu}]$$
$$= \underset{Y}{E}\Big[\underbrace{\underset{X|Y}{E}[e^{iXYu} | Y]}\Big]$$

Table $\boxed{\phi_X(u) = e^{-u^2/2}}$

donc $\varphi_Z(u) = E[e^{-\frac{u^2 Y^2}{2}}]$
$$= e^{-\frac{u^2}{2}(1)^2} \times \overbrace{P[Y=1]}^{p}$$
$$+ e^{-\frac{u^2}{2}(-1)^2} \times \underbrace{P[Y=-1]}_{q=1-p}$$

$$= e^{-u^2/2}$$

donc $\boxed{Z \sim N(0,1)}$

Example 2 $\quad Y_N = \sum_{i=1}^{N} X_i$

$X_i \to 1 \quad p$
$\quad \to 0 \quad q = 1-p$

$N \sim P(\lambda) \qquad P[N=k] = \frac{\lambda^k}{k!} e^{-\lambda} \quad k \in \mathbb{N}$

$$E[Y_N] = E_N\Big[\underbrace{E[Y_N \mid N]}_{\sum_{i=1}^{N} \underbrace{E[X_i]}_{p} = Np}\Big] = E_N[Np]$$

$$E[Y_N] = p\,E(N) \underset{\text{Table}}{=} \boxed{p\lambda}$$

Changements de variables

en 1D.

$X \sim N(0,1)$ $\qquad p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $x \in \mathbb{R}$

Loi de $Y = \frac{1}{X}$?

$Y = \frac{1}{X} \Leftrightarrow X = \frac{1}{Y}$

Densité de Y $\qquad \pi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2y^2}} \underbrace{\left|\frac{-1}{y^2}\right|}_{\frac{dx}{dy}}$ $\qquad y \in \mathbb{R}^*$

Preuve. (continu)

$\begin{pmatrix} U \\ V \end{pmatrix} = g(X,Y)$ $\qquad$ densité de $\begin{pmatrix} U \\ V \end{pmatrix}$?

$$P\left[\begin{pmatrix} U \\ V \end{pmatrix} \in \Delta\right] = \iint_\Delta \pi(u,v)\,du\,dv \qquad \forall \Delta$$

$$P\left[\begin{pmatrix} U \\ V \end{pmatrix} \in \Delta\right] = P[g(X,Y) \in \Delta]$$

$$= P[(X,Y) \in g^{-1}(\Delta)] = \qquad \forall \Delta$$

g bijection

$$= \iint_{g^{-1}(\Delta)} p(x,y)\,dx\,dy$$

$$\begin{pmatrix} U \\ V \end{pmatrix} = g(X,Y)$$

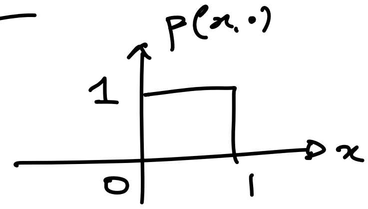
$$= \iint_{\Delta} p\left[g^{-1}(u,v)\right] \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| du\,dv$$

Égalité $\forall \Delta$

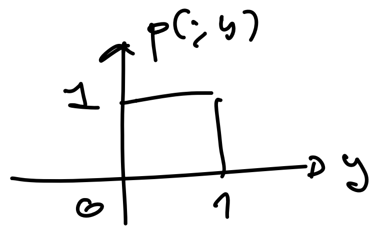
donc $\boxed{\pi(u,v) = p\left[g^{-1}(u,v)\right] \left|\det J\right|}$

Exemple 1

$X \sim U(0,1)$



$Y \sim U(0,1)$



X et Y indépendants

$$\Rightarrow p(x,y) = \begin{cases} 1 & \text{si } (x,y) \in \,]0,1[\times]0,1[\\ 0 & \text{sinon} \end{cases}$$

Loi de $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} X+Y \\ X \end{pmatrix}$?

Densité de $\begin{pmatrix} U \\ V \end{pmatrix}$

$$\pi(u,v) = \frac{1}{p\left[g^{-1}(u,v)\right]} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\begin{cases} U = X+Y \\ V = X \end{cases} \Leftrightarrow \begin{cases} X = V \\ Y = U-V \end{cases}$$

donc le changement de variables est bijectif

Jacobien $J = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

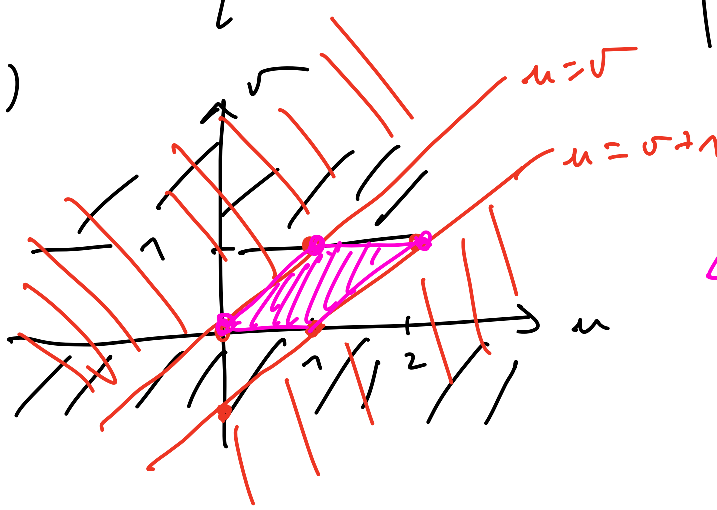
$$|\det(J)| = |-1| = 1$$

donc $\boxed{\pi(u,v) = 1 \quad (u,v) \in \Delta ?}$

Domaine de $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} X+Y \\ X \end{pmatrix}$

$$\begin{cases} 0 < x < 1 \\ 0 < y < 1 \end{cases} \Leftrightarrow \begin{cases} 0 < v < 1 \\ 0 < u-v < 1 \end{cases} \Leftrightarrow \boxed{\begin{cases} v > 0 \\ v < 1 \\ u > v \\ u < v+1 \end{cases}}$$

Domaine de (X,Y)



$\Delta =$

Exemple 2

$X \sim N(0,1)$ $\quad p(x,\cdot) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $\quad x \in \mathbb{R}$

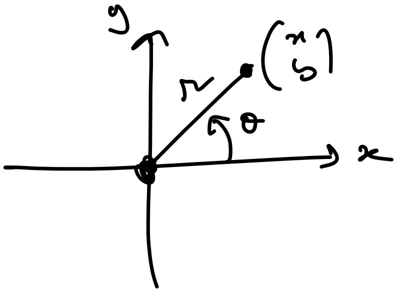
$Y \sim N(0,1)$ $\quad p(\cdot,y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ $\quad y \in \mathbb{R}$

X et Y indépendants

Quelle est la loi de $\begin{pmatrix} R \\ \theta \end{pmatrix}$ avec $\begin{cases} X = R\cos\theta \\ Y = R\sin\theta \end{cases}$

Loi de (X,Y)

$$p(x,y) = p(x,\cdot)\,p(\cdot,y) = \frac{1}{2\pi}\exp\left(-\frac{1}{2}(x^2+y^2)\right)$$
$$(x,y) \in \mathbb{R}^2$$



Le changement de variable est bijectif de $\mathbb{R}^2 \setminus \{(0,0)\}$ dans $\underbrace{]0,+\infty[}_{R} \times \underbrace{[0,2\pi[}_{\theta}$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \iff \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \mathrm{Atan}\left(\frac{y}{x}\right) + k\pi \end{cases}$$

Densité de $\begin{pmatrix} R \\ \theta \end{pmatrix}$

$$\pi(r,\theta) = \frac{1}{2\pi}\exp\left(-\frac{r^2}{2}\right)\left|\det(J)\right|$$

$x^2+y^2 = r^2$

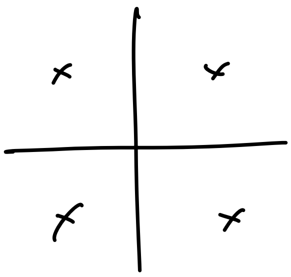
$$J = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$\left|\det(J)\right| = r$$

$$\boxed{\pi(r,\theta) = \frac{r}{2\pi}\exp\left(-\frac{r^2}{2}\right) \quad (r,\theta) \in]0,+\infty[\times]0,2\pi[}$$

$$\begin{cases} U = X^2 \\ V = Y^2 \end{cases} \Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \sqrt{U} \\ \sqrt{V} \end{pmatrix} \text{ ou } \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \sqrt{U} \\ -\sqrt{V} \end{pmatrix}$$

$$\text{ou } \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -\sqrt{U} \\ -\sqrt{V} \end{pmatrix} \text{ ou } \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -\sqrt{U} \\ \sqrt{V} \end{pmatrix}$$



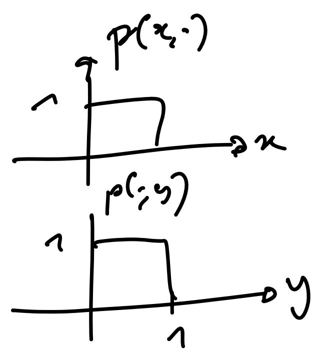
Exemple 3

$X \sim U(0,1)$

$Y \sim U(0,1)$

X et Y ind

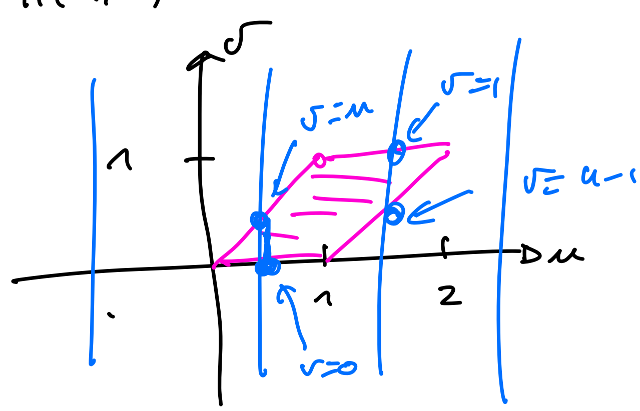
Loi de $U = X+Y$?



$$\begin{cases} U = X+Y \\ V = X \end{cases} \Longrightarrow$$

changement de variable habituel

$\pi(u,v) = 1$ si $(u,v) \in \Delta$

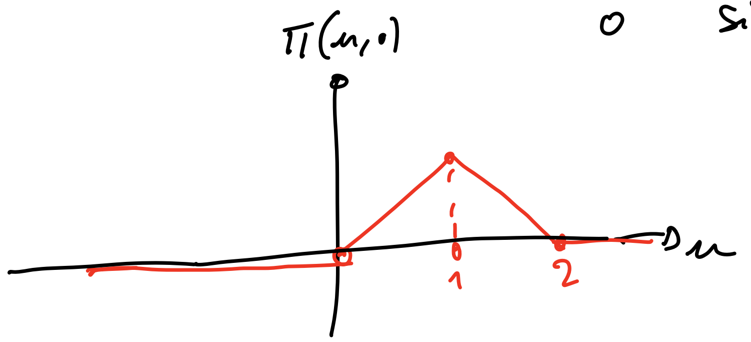


Loi de U

$$\pi(u,\cdot) = \int_{\mathbb{R}} \pi(u,v)\,dv$$

$$= \begin{cases} 0 & u<0 \\ 0 & u>2 \\ \displaystyle\int_0^u 1\,dv & u\in\,]0,1[\\ \displaystyle\int_{u-1}^1 1\,dv & u\in\,]1,2[\end{cases}$$

donc $\Pi(u,\cdot) = \begin{cases} u & \text{si } u\in\,]0,1[\\ 2-u & \text{si } u\in\,]1,2[\\ 0 & \text{sinon} \end{cases}$



$$\boxed{U = X+Y}$$

Exemple

$$\boxed{\begin{array}{ll} X \sim N(m_1, \sigma_1^2) & X \text{ et } Y \text{ ind} \\ Y \sim N(m_2, \sigma_2^2) & \\ \text{Loi de } U = X+Y \,? & \end{array}}$$

on peut calculer la fonction caractéristique de U

$$\phi_{X+Y}(t) = E\left[e^{iUt}\right] = E\Big[\underbrace{e^{i(X+Y)t}}_{e^{iXt}\, e^{iYt}}\Big]$$

$$= E\left[e^{iXt}\right] E\left[e^{iYt}\right]$$

$= \quad E[e^{ixt}]\, E[e^{iyt}]$

X et Y i.i.d $\Rightarrow e^{ixt}$ et e^{iyt} i.i.d

$\exp(i m_2 t - \frac{\sigma_2^2}{2} t^2)$ (Table)

$$\boxed{\phi_{X+Y}(t) = \phi_X(t)\, \phi_Y(t)}$$

$\exp(i m_1 t - \frac{\sigma_1^2}{2} t^2)$ (table)

$$\Rightarrow \phi_U(t) = \exp\Big(i \underbrace{(m_1 + m_2)}_{m} t - \frac{1}{2} \underbrace{(\sigma_1^2 + \sigma_2^2)}_{\sigma^2} t^2\Big)$$

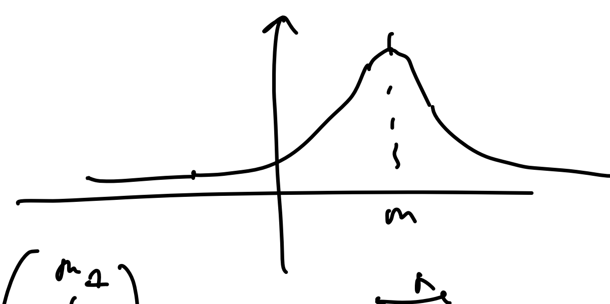
donc $U \sim N(m, \sigma^2)$

i.e $\boxed{U \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)}$

$\boxed{\text{Cours du } 3/10/2023}$

$\underline{\text{à 1D}}$ $\quad X \sim N(m, \sigma^2) \quad p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$

$x \in \mathbb{R}$



$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad m = \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \quad \exists\, \Sigma$

$(x - m)^T \Sigma^{-1} (x - m) = [x_1 - m_1, \dots, x_n - m_n] \begin{pmatrix} & \end{pmatrix}^{-1} \begin{pmatrix} x_1 - m_1 \\ \vdots \end{pmatrix}$

$$\mathbb{R}^n \quad \begin{pmatrix} x_n - m_n \end{pmatrix}$$

$n = 1$ $\qquad \Sigma = (\sigma^2)$ $\qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1n} \\ \Sigma_{n1} & & \Sigma_{nn} \end{pmatrix}$

$$\boxed{\begin{array}{lll} \Sigma \text{ symétrique} & \Sigma^T = \Sigma & \\ \Sigma \text{ positive} & x^T \Sigma x \geq 0 & \forall x \in \mathbb{R}^n \\ \Sigma \text{ définie} & x^T \Sigma x = 0 & \Rightarrow x = 0 \end{array}}$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 1/2 \\ 1 & 1/2 & 4 \end{pmatrix}$$

après diagonalisation

$$\Downarrow$$

$$\begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{pmatrix}$$

si $n = 1$, on retombe sur la loi normale $N_1(m, \sigma^2)$

si Σ diagonale, i.e., $\qquad \Sigma = \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{pmatrix}$

$$\det(\Sigma) = \prod \sigma_i^2$$

$$\Sigma^{-1} = \begin{pmatrix} 1/\sigma_1^2 & & \\ & \ddots & \\ & & 1/\sigma_n^2 \end{pmatrix} \begin{pmatrix} x_1 - m_1 \\ \vdots \\ x_n - m_n \end{pmatrix}$$

$$(x-m)^T \Sigma^{-1} (x-m) = [x_1 - m_1, \ldots, x_n - m_n] \begin{pmatrix} \frac{x_1 - m_1}{\sigma_1^2} \\ \vdots \\ \frac{x_n - m_n}{\sigma_n^2} \end{pmatrix}$$

$$= \sum_{i=1}^{n} \frac{(x_i - m_i)^2}{\sigma_i^2}$$

$$= \exp\left(-\frac{1}{2} \sum_{}^{n} \cdots\right)$$

$$\exp\left[-\frac{1}{2}(x-m)^T \Sigma^{-1}(x-m)\right] = \prod_{i=1}^{n} \exp\left[-\frac{1}{2\sigma_i^2}(x_i-m_i)^2\right]$$

$$p(x) = \frac{1}{(\sqrt{2\pi})^n}\,\frac{1}{\sqrt{\sigma_1^2 \dots \sigma_n^2}}\prod_{i=1}^{n} \exp\left[-\frac{1}{2\sigma_i^2}(x_i-m_i)^2\right]$$

$$p(x) = \prod_{i=1}^{n} \underbrace{\frac{1}{\sqrt{2\pi\sigma_i^2}}\exp\left[-\frac{(x_i-m_i)^2}{2\sigma_i^2}\right]}_{N(m_i,\,\sigma_i^2)}$$

<u>à 1D</u>

$$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\underbrace{-\frac{1}{2\sigma^2}(x-m)^2}_{-q(x)}\right]$$

$$q'(x) = \frac{1}{2\sigma^2}\,2(x-m) = \frac{x-m}{\sigma^2}$$

$$q''(x) = \frac{1}{\sigma^2}$$

$$K \exp(-x^2+2x)$$

$$q(x) = x^2-2x$$

$$q'(x) = 2x-2 = \frac{x-m}{\sigma^2}$$

$$q''(x) = 2 \qquad \frac{1}{\sigma^2} = 2 \Rightarrow \boxed{\sigma^2 = \frac{1}{2}}$$

$$x-m = \sigma^2(2x-2) = \frac{1}{2}(2x-2) = x-1$$

$\boxed{m=1}$

à n dimensions, on montre que $\quad q(x_1, \dots, x_n)$

$$q'(x) = \begin{pmatrix} \frac{\partial q}{\partial x_1} \\ \\ \frac{\partial q}{\partial x_n} \end{pmatrix}$$

$$q''(x) = \begin{pmatrix} \frac{\partial^2 q}{\partial x_1^2} & \cdots & \frac{\partial^2 q}{\partial x_1 \partial x_n} \\ & & \\ \frac{\partial^2 q}{\partial x_n \partial x_1} & & \frac{\partial^2 q}{\partial x_n^2} \end{pmatrix}$$

On montre que si $X \sim N(m, \sigma^2)$ alors

$$q''(x) = \Sigma^{-1} \qquad \left(\frac{1}{\sigma^2} \text{ à } 1D \right)$$

$$q'(x) = \Sigma^{-1}(x-m) \qquad \left(\frac{x-m}{\sigma^2} \text{ à } 1D \right)$$

Exemple $\quad p(x,y) \propto \exp\left[\underbrace{-x^2 - \frac{3}{2}y^2 - xy + 4x + 7y}_{q(v)} \right] \quad v = \begin{pmatrix} x \\ y \end{pmatrix}$

$$q(v) = x^2 + \frac{3}{2}y^2 + xy - 4x - 7y$$

$$q''(v) = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = \Sigma^{-1} = U$$

donc $U^{-1} = \frac{1}{\det U} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}^T = \Sigma$

$\det(U) = 3 \times 2 - 1 = 5$

$$\boxed{\Sigma = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}}$$

Recherche de m

$q'(v) = \Sigma^{-1}(x - m) \qquad v = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 2x + y - 4 \\ 3y + x - 7 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x - m_1 \\ y - m_2 \end{pmatrix}$$

$$\begin{cases} 2\cancel{x} + \cancel{y} - 4 = 2(\cancel{x} - m_1) + \cancel{y} - m_2 \\ 3\cancel{y} + \cancel{x} - 7 = \cancel{x} - m_1 + 3(\cancel{y} - m_2) \end{cases}$$

$$\begin{cases} +2m_1 + m_2 = +4 & (1) \\ +m_1 + 3m_2 = +7 & (2) \end{cases}$$

système de 2 équations à 2 inconnus

$(1) - 2 \times (2) \Rightarrow 2\cancel{m_1} + m_2 - 2(\cancel{m_1} + 3m_2) =$

$4 - 2(7)$

$-5m_2 = -10 \Rightarrow \boxed{m_2 = 2}$

$3 \times ① - ② \Rightarrow 6m_1 - m_1 = 12 - 7$

$5m_1 = 5 \qquad \boxed{m_1 = 1}$

Résumé

$$m = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix}$$

matrice de covariance

$$\begin{pmatrix} \mathrm{Var}\, X & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(X,Y) & \mathrm{Var}\, Y \end{pmatrix}$$

$$\phi(u) = E\left[e^{iu^T X}\right] = E\left[e^{i(u_1 X_1 + \dots + u_n X_n)}\right]$$

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$= \int\int \cdots \int e^{i(u_1 x_1 + \dots + u_n x_n)} p(x_1 \dots x_n)\, dx_1 \dots dx_n$$

$= \dots$ (30 minutes)

$$= \exp\left(i u^T m - \frac{1}{2} u^T \Sigma u\right)$$

$$\frac{\partial \phi}{\partial u_1} = E\left[e^{i(u_1 x_1 + \dots + u_n x_n)} \times i x_1 \right]$$

$$\left.\frac{\partial \phi}{\partial u_1}\right|_{u=0} = i\,E[x_1]$$

<u>Rappel (1D)</u> si $X \sim N(m, \sigma^2)$

alors $\underbrace{2X+3}$ $\sim \boxed{N(2m+3,\ 4\sigma^2)}$

<u>Transformation affine à n dimension</u>

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_p \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{p1} & \dots & a_{pn} \end{pmatrix} X + \begin{pmatrix} b_1 \\ \vdots \\ b_p \end{pmatrix}}_{AX+b}$$

<u>EX</u> $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \qquad Y = \begin{pmatrix} 2X_1 + X_2 - 1 \\ X_1 - X_3 \\ X_1 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$

$E(Y)$? Σ_Y?

$$E(Y) = \begin{pmatrix} E(Y_1) \\ E(Y_2) \end{pmatrix} \qquad \Sigma_Y = \begin{pmatrix} \mathrm{Var}\, Y_1 & \mathrm{cov}(Y_1, Y_2) \\ \mathrm{cov}(Y_1, Y_2) & \mathrm{Var}\, Y_2 \end{pmatrix}$$

Mais $E(Y_1) = E[2X_1 + X_2 - 1] = 2E[X_1] + E[X_2] - 1$

$E(Y_2) = E(X_1 - X_3) = E[X_1] - E[X_3]$

donc $E(Y) = \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} E(X_1) \\ E(X_2) \\ E(X_3) \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Si $\boxed{Y = AX + b \text{ alors } E(Y) = AE(X) + b}$

$\boxed{\Sigma_Y = A \Sigma_X A^T}$

Exemple d'application

$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{m}, \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{\Sigma} \right)$

$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 2X_1 + X_2 - 1 \\ X_1 - X_3 \end{pmatrix}$

Quelle est la loi de Y ?

$\underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{b}$

donc $E[Y] = A\,E[X] + b$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ -2 \end{pmatrix}}$$

$$\Sigma_Y = A\,\Sigma_X\,A^T$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\boxed{\Sigma_Y = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}}$$

<u>Preuve</u> $\quad Y = AX + B \qquad X \sim N(m, \Sigma)$

$$\Phi_Y(u) = E\left[e^{u^T Y}\right] = E\left[e^{u^T A X + u^T B}\right]$$

$$= E\left[e^{u^T A X}\, e^{u^T B}\right]$$

$$= e^{u^T B} \; E\big[e^{\underbrace{u^T A x}_{v^T x}} \big]$$

$$\boxed{v = A^T u}$$

$$= e^{u^T B} \; \Phi_X(A^T u)$$

$$X \sim N(m, \Sigma) \Rightarrow \Phi_X(u) = \exp\left[u^T m + \frac{1}{2} u^T \Sigma u \right]$$

On en déduit

$$\Phi_Y(u) = e^{u^T B} \exp\left[u^T A m + \frac{1}{2} u^T A \Sigma A^T u \right]$$

$$= \exp\left[u^T (\underbrace{Am + B}_{m_Y}) + \frac{1}{2} u^T (\underbrace{A \Sigma A^T}_{\Sigma_Y}) u \right]$$

il faut quand même s'assurer que $A \Sigma A^T$ est symétrique définie positive

$$(A \Sigma A^T)^T = A \underbrace{\Sigma^T}_{} A^T \underset{\substack{\uparrow \\ X \sim N(m, \Sigma)}}{=} A \Sigma A^T =$$

donc Σ_Y symétrique

$$\underbrace{y^T (A \Sigma A^T) y}_{} \geq 0 \,!$$

$(y^T A) \Sigma (A^T y) \geq 0$ car $\underline{\Sigma \text{ positive}}$

x^T $\qquad\qquad$ x

Σ_y définie ?

$$y^T (A \Sigma A^T) y = 0 \Rightarrow y = 0$$

$$\Downarrow$$

$$(y^T A) \Sigma (A^T y) = 0$$

$$\Downarrow \ \Sigma \text{ définie}$$

$$A^T y = 0$$

$$\Downarrow \quad \text{si } \boxed{rg(A) = p}$$

$$y = 0$$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

ici $rg A = 2$ car les 2 lignes ne sont pas colinéaires.

$$\boxed{\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N(m, \Sigma) \qquad \text{Loi de } \begin{pmatrix} X_1 \\ X_3 \end{pmatrix} ?}$$

$$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{B}$$

$\operatorname{rg} A = 2$ donc $\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N(\, ? \,,\, ? \,)$

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 6 \end{pmatrix} \right)$$

$E[X_1]$ $\quad$ $E(X_3)$ $\quad$ $\operatorname{var} X_1$ $\quad$ $\operatorname{cov}(X_1, X_3)$ $\quad$ $\operatorname{var} X_3$

donc $\boxed{\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 & 2 \\ 2 & 6 \end{pmatrix} \right)}$

si $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ vecteur Gaussien et $\operatorname{cov}(X_1, X_2) = 0$

$\Rightarrow$ X_1 et X_2 sont indépendants

TD du 9/10

$X \sim N(0,1)$ $\Rightarrow$ $E[X] = 0$, $\operatorname{var}(X) = 1$

$Y \sim N(0,1)$

$Z \sim N(0,1)$

X, Y et Z sont indépendantes donc $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ est un vecteur gaussien de vecteur moyenne $m = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ et de matrice de covariance $\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$U = (1\ 1\ 1)\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ transformation affine d'un vecteur gaussien $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} + B$

$A = [1\ \ 1\ \ 1]\quad B = 0$

$rg(A) = 1 \Rightarrow U \sim N(Am + B,\ A\Sigma A^T)$

$(1\ 1\ 1)\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + 0 = 0 \qquad \underbrace{(1\ 1\ 1) I \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{3}$

$$\boxed{U \sim N(0,3)}$$

Rq : $E[e^{itU}] = E[e^{it(X+Y+Z)}]$

$= E(e^{itX}\ e^{itY}\ e^{itZ})$

X, Y, Z ind

$= \underbrace{\phi_X(t)}\ \phi_Y(t)\ \phi_Z(t)$

? ? ?

$e^{-t^2/2}\quad e^{-t^2/2}\quad e^{-t^2/2}$

$= \exp\left(-3\frac{t^2}{2}\right)$ $\quad m = 0,\ \sigma^2 = 3$

Table $X \sim N(m, \sigma^2)$ $\quad \phi_X(t) = \exp\left(imt - \frac{\sigma^2 t^2}{2}\right)$

donc $\boxed{U \sim N(0,3)}$

2) $\boxed{\text{Montrer que } T = X - Y \text{ est indépendante de } U = X+Y+Z}$

$$\begin{pmatrix} T \\ U \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{B}$$

$rg\,A = 2$ donc $\begin{pmatrix} T \\ U \end{pmatrix}$ est un vecteur gaussien

de $N(\underbrace{Am + B}_{0},\ \underbrace{A \Sigma A^T}_{})$

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} I \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}}$$

$\begin{pmatrix} T \\ U \end{pmatrix}$ vecteur gaussien

$cov(T, U) = 0$ $\Rightarrow$ T et U indépendantes

indépendance $\Rightarrow$ covariance $= 0$

$\Leftarrow$ dans le cas d'un vecteur gaussien

Exo1

1) $X_1 \sim N(0, 1)$

$X_2 \sim N(0, 1)$

$X_3 \sim N(0, 1)$

X_1, X_2 et X_3 indépendants

donc $V = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ est un vecteur gaussien de moyenne $E[V] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} \\ 0 \end{pmatrix}$

et de matrice de covariance

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

$$p(x_1, x_2, x_3) = p(x_1)\, p(x_2)\, p(x_3)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2} \frac{1}{\sqrt{2\pi}} e^{-x_3^2/2}$$

$$= \boxed{\frac{1}{(\sqrt{2\pi})^3} \exp\left[-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\right]}$$

2) $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = P^T \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ $\qquad P^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$

$\boxed{\text{lois de } Y, \quad Y_1, Y_2 \text{ et } Y_3}$

P^T matrice orthogonale

donc $(P^T)^T = P = (P^T)^{-1}$

$\Rightarrow \det(P^T) \neq 0$

$\boxed{rg(P^T) = 3}$

$rg(P^T) = 3$

$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ vecteur gaussien $\quad \Rightarrow Y = P^T X \sim N\Big(\underbrace{P^T m}_{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}, P^T \Sigma P\Big)$

$$P^T \underbrace{\Sigma}_{I} P = P^T P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

donc $\boxed{Y \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, I_3\right)}$

2) Montrer que $\sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3} y_i^2$ et que $y_2^2+y_3^2 = \sum_{i=1}^{3} x_i^2 - 3\bar{x}^2$

avec $\bar{x} = \frac{1}{3}\sum_{i=1}^{3} x_i$

- $\boxed{\sum_{i=1}^{3} y_i^2} = y^T y = (P^T x)^T P^T x = x^T \underbrace{P P^T}_{I} x = x^T x = \boxed{\sum_{i=1}^{3} x_i^2}$

- $\boxed{y_2^2+y_3^2} = y_1^2+y_2^2+y_3^2 - y_1^2$

$= \sum_{i=1}^{3} x_i^2 - y_1^2$

Mais $y = P^T x = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ \multicolumn{3}{c}{\rule{3cm}{0.4pt}} \\ \multicolumn{3}{c}{\rule{3cm}{0.4pt}} \end{pmatrix} x$

donc $y_1 = \frac{1}{\sqrt{3}} x_1 + \frac{1}{\sqrt{3}} x_2 + \frac{1}{\sqrt{3}} x_3$

$= \frac{1}{\sqrt{3}} \Big(\underbrace{\frac{x_1+x_2+x_3}{3}}_{\bar{x}} \Big) \times 3 = \boxed{\sqrt{3}\,\bar{x} = y_1}$

d'où $\boxed{y_2^2+y_3^2 = \sum_{i=1}^{3} x_i^2 - 3(\bar{x})^2}$

$\bar{x} = \frac{1}{3}\sum x_i$

3) Exprimer $\bar{x}$ et s^2 en fonction des variables y_i

$\boxed{\bar{x} = \frac{y_1}{\sqrt{3}}}$

$s^2 = \frac{1}{2}\sum_{i=1}^{3} (x_i - \bar{x})^2 = \frac{1}{2}\left(\sum_{i=1}^{3} x_i^2 - 2\bar{x} \sum_{i=1}^{3} x_i + \sum_{i=1}^{3} \bar{x}^2 \right)$

$= \frac{1}{2}\left(y_1^2+y_2^2+y_3^2 - 2\frac{y_1}{\sqrt{3}} \times 3 \frac{y_1}{\sqrt{3}} + 3\frac{y_1^2}{3} \right)$

$\boxed{s^2 = \frac{y_2^2+y_3^2}{2}}$

Y_1, Y_2 et Y_3 ind $\Rightarrow$ $\boxed{\bar{X} \text{ et } S^2 \text{ sont des va indépendantes !!}}$

4) Lois de $\bar{X}$ et de $2S^2$

$$\bar{X} = \frac{1}{3}\sum_{i=1}^{3} X_i = \frac{Y_1}{\sqrt{3}} \sim N\left(0, \frac{\operatorname{Var} Y_1}{3}\right) = \boxed{N\left(0, \frac{1}{3}\right)}$$

$$\boxed{2S^2 = Y_2^2 + Y_3^2 \sim \chi^2_2}$$

Exercice 3

$$X \sim N(0,1)$$

$$1 + X \sim N(1,1)$$

$$aX + b \quad \sim N(1, 10)$$

$$E[aX+b] = 1 \Rightarrow b = 1$$

$$\operatorname{Var}(aX+b) = a^2 \underbrace{\operatorname{Var} X}_{1} = 10$$

$$a = \sqrt{10}$$

$$\boxed{\sqrt{10}\, X + 1} \sim N(1, 10)$$

$$Y \sim N(\mu, \Sigma) \ ?$$

$$X \sim N(0, I)$$

$$AX + B \sim N(B, A \overset{I}{\Sigma} A^T)$$

Ex $\quad Y \sim N\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix}\right) \qquad B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$(2 \ 1 \ 1)$

$$AA^T = \underbrace{\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix}}_{M}$$

routine MATLAB chol.m

$\underset{\text{entrée}}{M} \Rightarrow A$

Cours du 10/10/2023

$$X \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 & 1 \\ 0 & 10 & 2 & 3 \\ 0 & 2 & 5 & 1 \\ 1 & 3 & 1 & 7 \end{pmatrix} \right)$$

$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}$

$Cov(X_1, X_2) = 0$
$Cov(X_1, X_3) = 0$ $\Longrightarrow$ X_1 et X_2 ind, X_1 et X_3 ind, X_1 et $\begin{pmatrix} X_2 \\ X_3 \end{pmatrix}$ ind
X vecteur gaussien

$$\boxed{f(x) = \frac{1}{(2\pi)^{n/2} \sqrt{\det \Sigma}} \exp\left[-\frac{1}{2} (x-m)^T \Sigma^{-1} (x-m) \right]}$$

si $m = 0$ $\quad \Sigma = \begin{pmatrix} \Sigma' & 0 \\ 0 & \Sigma'' \end{pmatrix}$ $\quad \Sigma^{-1} = \begin{pmatrix} (\Sigma')^{-1} & 0 \\ 0 & (\Sigma'')^{-1} \end{pmatrix}$

$\det(\Sigma) = \det(\Sigma') \det(\Sigma'')$

$$(x-m)^T \Sigma^{-1} (x-m) = \begin{bmatrix} x' - m' \\ x'' - m'' \end{bmatrix}^T \begin{bmatrix} (\Sigma')^{-1} & 0 \\ 0 & (\Sigma'')^{-1} \end{bmatrix} \begin{bmatrix} x' - m' \\ x'' - m'' \end{bmatrix}$$

$$= (x'-m')^T (\Sigma')^{-1} (x'-m') + (x''-m'')^T (\Sigma'')^{-1} (x''-m'')$$

donc $f(x) = f(x', \cdot)\, f(\cdot, x'')$ donc X' et X'' sont des vecteurs indépendants

$X \sim N(m, \Sigma)$ alors $Y = AX + b \sim N(\underbrace{Am+b}_{m_Y}, A\Sigma A^T)$

$\operatorname{rg} A = p$

?

$$\Sigma_Y = E\left((Y - m_Y)(Y - m_Y)^T \right)$$

si $m_Y = 0$ $\quad = E[Y Y^T]$

$$= E(A X X^T A^T) = A \underbrace{E(X X^T)}_{\Sigma_X} A^T$$

$$\boxed{X_1, \dots, X_n \xrightarrow[n \to +\infty]{} X \ ?}$$

$$X_n \begin{cases} \to 1 \\ \to 0 \end{cases}$$

$$P(X_n = 1) = \frac{1}{n}$$

$$P(X_n = 0) = 1 - \frac{1}{n}$$

$$\boxed{X_n \xrightarrow[n \to +\infty]{\mathcal{L}} X = 0 \ ?}$$

$$\varphi_n(t) = E[e^{itX_n}] = e^{it} P(X_n = 1) + e^{it \times 0} P(X_n = 0)$$

$$= \frac{1}{n} e^{it} + 1 - \frac{1}{n} \xrightarrow[n \to +\infty]{} 1$$

$$\varphi(t) = E[e^{itX}]$$

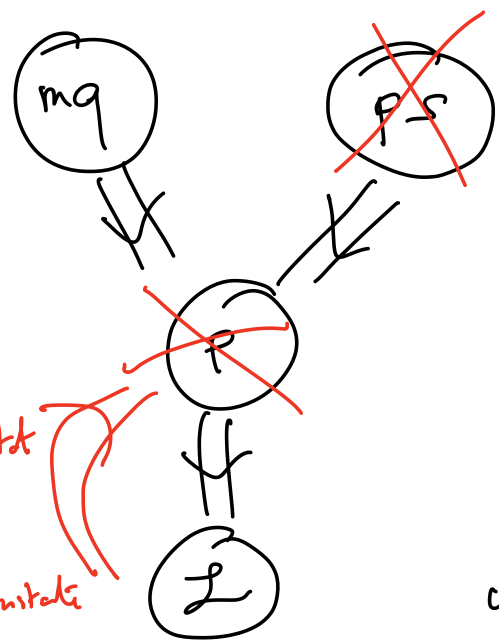
$$= E[e^{it \times 0}] = E(1) = 1$$

$$X_n \begin{cases} \to 1 & \frac{1}{n} \\ \to 0 & 1-\frac{1}{n} \end{cases} \qquad X_n \xrightarrow[n\to+\infty]{m.q} X=0 \ ?$$

$$E\left((X_n - X)^2\right) \underset{\substack{\uparrow \\ X=0}}{=} E(X_n^2) = 1^2 \times \frac{1}{n} + 0^2 \times \left(1-\frac{1}{n}\right)$$

$$= \frac{1}{n} \xrightarrow[n\to+\infty]{} 0$$

donc $\boxed{X_n \xrightarrow[m.q]{n\to+\infty} X=0}$



convergences fortes

si $X_n \xrightarrow{\mathcal{L}} X =$ constante

$\Downarrow$

$X_n \xrightarrow{P} X =$ constante

convergence faible

$$E\left[(\overline{X}-m)^2\right] = E\left[\left(\frac{1}{n}\sum_{k=1}^{n} x_k - m\right)^2\right]$$

$$= E\left[\left(\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} m\right)^2\right]$$

$$= \frac{1}{n^2} \ E\left[\sum_{k=1}^{n}(X_k - m) \sum_{l=1}^{n}(X_l - m)\right]$$

$$(a_1+a_2 \dots + a_n)(a_1+a_2+\dots+a_n)$$

$$\sum_i \sum_j a_i a_j$$

$$= \frac{1}{n^2} E\left[\sum_{k,l} (X_k - m)(X_l - m) \right]$$

$$= \frac{1}{n^2} \sum_{k,l} E\left[(X_k - m)(X_l - m)\right]$$

→ $\underbrace{Var(X_k)}_{=\sigma^2}$ si $k = l$

→ $\underbrace{cov(X_k, X_l)}_{=0}$ si $k \neq l$

car les variables sont ind

$$= \frac{1}{n^2} \times n\sigma^2 = \boxed{\frac{\sigma^2}{n}} \xrightarrow[n \to +\infty]{} 0$$

Théorème de la limite centrale

Loi de $\sum_{k=1}^{n} X_k$ ou de $\frac{1}{n}\sum_{k=1}^{n} X_k$

lorsque $n \to +\infty$

$$E\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} \underbrace{E(X_k)}_{m} = nm$$

$$Var\left[\sum_{k=1}^{n} X_k\right] = \sum_{k=1}^{n} \underbrace{Var(X_k)}_{\sigma^2} = n\sigma^2$$

$X_1 \dots X_n$ ind

$$\sum_{k=1}^{n} X_k \approx N(nm, n\sigma^2)$$

$$\cancel{\sum_{k=1}^{n} X_k \xrightarrow[n\to+\infty]{} \underbrace{N(nm, n\sigma^2)}}$$ HORRIBLE !

$$\boxed{\frac{\sum_{k=1}^{n} X_k - nm}{\sqrt{n\sigma^2}} \xrightarrow[n\to+\infty]{} N(0,1)}$$

$$\boxed{\frac{\frac{1}{n}\sum_{k=1}^{n} X_k - m}{\sqrt{\frac{\sigma^2}{n}}} \xrightarrow[n\to+\infty]{} N(0,1)}$$

<u>Remarque</u> : Si X et Y sont des variables aléatoires indépendantes, alors $\mathrm{Var}(X+Y) = \mathrm{Var}\,X + \mathrm{Var}\,Y$

$$
\begin{aligned}
\mathrm{Var}(X+Y) &= E[(X+Y)^2] - E^2(X+Y) \\
&= E(X^2+Y^2+2XY) - \underbrace{(E(X)+E(Y))^2}_{E^2(X)+E^2(Y)+2E(X)E(Y)} \\
&= E(X^2) - E^2(X) + E(Y^2) - E^2(Y) + 2\left(E(XY) - E(X)E(Y)\right)
\end{aligned}
$$

$$\boxed{Var(X+Y) = Var X + Var Y + 2\, cov(X,Y)}$$

Si X et Y ind, $\Rightarrow cov(X,Y) = 0 \Rightarrow \boxed{Var(X+Y) = Var X + Var Y}$

THE END ...