Exercice 1 — 10

1] $p(x,\cdot) = \theta^2 x e^{-\theta x}$ $x>0$ — 2pts

$p(\cdot,y) = \theta e^{-\theta y}$ $y>0$ — (4)

$X \sim \Gamma(\theta,2)$ $E[X] = \frac{2}{\theta}$ $Var X = \frac{2}{\theta^2}$ — 1pt

$Y \sim \Gamma(\theta,1)$ $E[Y] = \frac{1}{\theta}$ $Var Y = \frac{1}{\theta^2}$

$X \geq Y \Rightarrow X$ et Y ne sont pas ind — 1pt

2] $\begin{cases} X = T + Z \\ Y = T \end{cases}$ $|J| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$ $T \geq 0$, $Z \geq 0$ — 2pts

$g(z,t) = \theta^2 \exp[-\theta(z+t)]$ $z \geq 0$ $t \geq 0$ — (3)

$g(z,\cdot) = \theta e^{-\theta z}$ $z>0$

$Z^2 = U$ $q(u) = \frac{\theta}{2\sqrt{u}} e^{-\theta\sqrt{u}}$ $u>0$ — 1pt

3] Z et T ind — 1pt — (2)

$cov(X,Y) = Var Y = \frac{1}{\theta^2}$ — 1pt

4] $\phi_X(u) = \phi_Z(u)\phi_T(u) \Rightarrow X \sim \Gamma(\theta,2)$ — (1)

Exercice 2 — 6

1) $\hat{\theta}_{MV} = \frac{1}{n}\sum X_i^2$ — 1pt — (4)

$E[\hat{\theta}_{MV}] = \theta$ — 1pt

$Var\,\hat{\theta} = \frac{Var X_i^2}{n}$ — 1pt

$Var X_i^2 = 2\theta^2 = 2\theta^4$ 1pt

2) $BCR = \frac{2\theta^2}{n} = \frac{2\theta^4}{n}$ — 1pt — (1)

3) $\hat{\theta}_{MO} = \frac{1}{n}\sum X_i^2$ — (1)

Exercice 3 — 7

1) Rejet de H_0 si $\sum X_i < S_\alpha$ — 1pt — (1)

2) $T \sim P(n\lambda)$ 1pt — (2)

Fastidieux 1pt

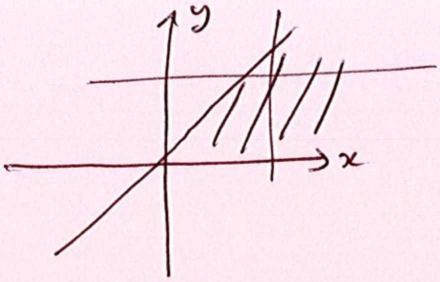
3) $T \sim N(n\lambda, n\lambda)$ — 1pt

$S_\alpha = \sqrt{n\lambda_0}\,\phi^{-1}(\alpha) + n\lambda_0$ — 1pt

$\Pi = \phi\left[\sqrt{\frac{\lambda_0}{\lambda_1}}\,\phi^{-1}(\alpha) + \sqrt{n}\,\frac{\lambda_0 - \lambda_1}{\sqrt{\lambda_1}}\right]$ 1pt — (4)

paramètres qui influent sur le test — 1pt

<u>Ex1</u>



1) $x>0$ $\quad p(x,\cdot) = \int_0^x \theta^2 e^{-\theta x}\, dy$

$p(x,\cdot) = \theta^2 x e^{-\theta x} \mathbb{1}_{\mathbb{R}^+}(x)$ i.e $X \sim \Gamma(\theta,2)$ $\quad \nu=2$

$p(\cdot,y) = \int_y^{+\infty} \theta^2 e^{-\theta x}\, dx = \left[-\theta e^{-\theta x}\right]_y^{+\infty} = \theta e^{-\theta y} \mathbb{1}_{\mathbb{R}^+}(y)$

$Y \sim \Gamma(\theta,1)$ $\quad \nu=1$

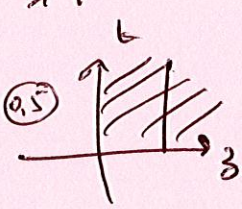
$E[X] = \frac{\nu}{\theta} = \frac{2}{\theta}$ $\qquad E[Y] = \frac{1}{\theta}$

$Var\, X = \frac{2}{\theta^2}$ $\qquad Var\, Y = \frac{1}{\theta^2}$

$x \geq y \Rightarrow$ les va X et Y ne sont pas indépendantes

2) $\begin{cases} Z = X - Y \\ T = Y \end{cases} \iff \begin{cases} Y = T \\ X = Z + T \end{cases}$ (0,5) $\qquad J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$ (0,5)

Domaine $\quad x \geq y \geq 0 \iff z + t \geq t \geq 0 \iff \begin{cases} t \geq 0 \\ z \geq 0 \end{cases}$ (0,5)



$$\boxed{g(z,t) = \theta^2 e^{-\theta(z+t)} \mathbb{1}_\Delta(z,t)}$$ (0,5)

(2 pts)

<u>Loi de Z</u> $\quad g(z,\cdot) = \int_0^{+\infty} \theta^2 e^{-\theta z} e^{-\theta t}\, dt = e^{-\theta z}\left[-\theta e^{-\theta t}\right]_0^{+\infty}$

$g(z,\cdot) = \theta e^{-\theta z} \mathbb{1}_{\mathbb{R}^+}(z)$ $\qquad Z \sim \Gamma(\theta,1)$ (1)

<u>Loi de Z^2</u> $\quad$ le changement de va est bijectif de $\mathbb{R}^+$ dans $\mathbb{R}^+$

$U = Z^2 \iff Z = \sqrt{U}$ $\qquad J = \frac{1}{2\sqrt{u}}$

$q(u) = \theta e^{-\theta\sqrt{u}} \times \frac{1}{2\sqrt{u}} = \boxed{\frac{\theta}{2\sqrt{u}} e^{-\theta\sqrt{u}} \mathbb{1}_{\mathbb{R}^+}(u)}$ (1)

Rq $\quad \int_0^{+\infty} \frac{\theta}{2\sqrt{u}} e^{-\theta\sqrt{u}}\, du = \int_0^{+\infty} \frac{\theta}{2\sqrt{u}} \frac{\theta}{2} \frac{\theta}{v} e^{-v} \frac{1}{\theta^2} 2v\, dv = \left[-e^{-v}\right]_0^{+\infty} = 1$

$v = \theta\sqrt{u}$

$u = \frac{v^2}{\theta^2}$ $\qquad \sqrt{u} = \frac{v}{\theta}$

$\frac{1}{\sqrt{u}} = \frac{\theta}{v}$

3) $g(z,t) = \theta^2 e^{-\theta(z+t)} \mathbb{1}_{(\mathbb{R}^+)^2}(z,t)$ $g(z,t) = g(z,\cdot) \times g(\cdot,t)$ $\forall z \forall t$ (2)

$g(z,\cdot) = \theta e^{-\theta z} \mathbb{1}_{\mathbb{R}^+}(z)$ donc Z et T sont indépendants (1)

$g(\cdot,t) = \theta e^{-\theta t} \mathbb{1}_{\mathbb{R}^+}(t)$

$cov(Z,T) = 0 \Rightarrow E[ZT] - E[Z]E[T] = 0$

$\Rightarrow E[(X-Y)Y] - (E[X] - E[Y])(E[Y]) = 0$

$\Rightarrow E[XY] - \cancel{E[Y^2]} - E(X)E(Y) + \cancel{E[Y^2]}$ $E[Y]^2$ $= 0$

$\Rightarrow$ $\boxed{cov(X,Y) = E[Y^2] - E[Y]^2 = Var Y = \frac{1}{\theta^2}}$ (1)

4) $\phi_X(u) = \phi_Z(u)\,\phi_T(u) = \dfrac{1}{1-\frac{it}{\theta}} \times \dfrac{1}{1-\frac{it}{\theta}} = \dfrac{1}{\left(1-\frac{it}{\theta}\right)^2}$ (1pt)

donc $\boxed{X \sim \Gamma(\theta, 2)}$ CQFD

Ex 2 1) $f(x_1 \dots x_n;\theta) = \dfrac{1}{(2\pi\theta)^{n/2}} \exp\left(-\dfrac{1}{2\theta}\sum_{i=1}^{n} x_i^2\right)$

$\ln f = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\theta - \frac{1}{2\theta}\sum x_i^2$

$\dfrac{\partial \ln f}{\partial\theta} = 0 \iff -\frac{n}{2}\frac{1}{\theta} + \frac{1}{2\theta^2}\sum x_i^2 = 0 \iff \theta = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

$\boxed{\hat{\theta}_{mv} = \frac{1}{n}\sum_{i=1}^{n} x_i^2}$ (1pt)

(1pt)

$E[\hat{\theta}_{mv}] = \sigma^2 = \theta$ $\hat{\theta}_{mv}$ sans biais

$Var\,\hat{\theta}_{mv} = \dfrac{Var(x_i^2)}{n}$ $\hat{\theta}_{mv}$ cv $Var\,X_i^2 = E[X^4] - E^2[X^2] = 2\sigma^4$

2) $\dfrac{\partial^2 \ln f}{\partial\theta^2} = \frac{n}{2}\frac{1}{\theta^2} - \frac{2}{2\theta^3}\sum x_i^2$ Borne + Var $\hat{\theta}$

$E[\ \ ] = \frac{n}{2\theta^2} - \frac{1}{\theta^3} n\theta = \frac{n}{2\theta^2} - \frac{2n}{2\theta^2} = -\frac{n}{2\theta^2}$ (2pts)

$\boxed{Bcr(\theta) = \frac{2\theta^2}{n}}$ $Var\,\hat{\theta} = \frac{2\theta^2}{n} \geq BCR \Rightarrow \hat{\theta}$ estimateur efficace ~~non convergent~~

3) $E[x_i] = 0$ $E[x_i^2] = \sigma^2 = \theta$ $\Rightarrow$ $\boxed{\hat{\theta}_{mo} = \frac{1}{n}\sum_{i=1}^{n} x_i^2}$ 1pt

Exercice 3

1) Rejet de H_0 si

$$\frac{\prod_{i=1}^{n} \frac{\lambda_1^{x_i}}{x_i!} e^{-\lambda_1}}{\prod_{i=1}^{n} \frac{\lambda_0^{x_i}}{x_i!} e^{-\lambda_0}} > k_\alpha$$

(1pt)

si $\left(\frac{\lambda_1}{\lambda_0}\right)^{\sum_{i=1}^{n} x_i} > k_\alpha \, e^{-n\lambda_0 + n\lambda_1}$

$H_0 : \lambda = \lambda_0$

$H_1 \quad \lambda = \lambda_1 < \lambda_0$

si $\sum x_i \, \ln\left(\frac{\lambda_1}{\lambda_0}\right) > S_\alpha$

$\lambda_1 < \lambda_0 \Rightarrow \ln(\;) < 0$

Rejet de H_0 si $\sum x_i < S_\alpha$

Region critique
$\{(x_1, \dots x_n) / \sum x_i < S_\alpha\}$

2) $T = \sum X_i \qquad \phi_T(t) = \prod_{i=1}^{n} \phi_{X_i}(t) = \exp\left[n\lambda\,(e^{it} - 1)\right]$

donc **$T \sim P(n\lambda)$** (1pt)

Fastidieux de determiner le seuil (1pt)

3) • $T \sim N(n\lambda, n\lambda)$ (1pt)

$$\alpha = P[\,T < S_\alpha \mid \lambda = \lambda_0] = P\Big[\underbrace{\frac{T - n\lambda_0}{\sqrt{n\lambda_0}}}_{N(0,1)} < \frac{S_\alpha - n\lambda_0}{\sqrt{n\lambda_0}}\Big] = F_{N(0,1)}\left(\frac{S_\alpha - n\lambda_0}{\sqrt{n\lambda_0}}\right)$$

$$= \phi\left(\frac{S_\alpha - n\lambda_0}{\sqrt{n\lambda_0}}\right)$$

• $\Rightarrow$ **$S_\alpha = \sqrt{n\lambda_0}\,\phi^{-1}(\alpha) + n\lambda_0$** (1pt) $\qquad \alpha =$

• courbes COR

$$\Pi = P[\,T < S_\alpha \mid \lambda = \lambda_1] = P\left(\frac{T - n\lambda_1}{\sqrt{\lambda_1 n}} < \frac{S_\alpha - n\lambda_1}{\sqrt{n\lambda_1}}\right] = \phi\left(\frac{S_\alpha - n\lambda_1}{\sqrt{n\lambda_1}}\right)$$

$$\Pi = \phi\left[\frac{\sqrt{n\lambda_0}\,\phi^{-1}(\alpha) + n\lambda_0 - n\lambda_1}{\sqrt{n\lambda_1}}\right]$$

$$\boxed{\Pi = \phi\left[\sqrt{\frac{\lambda_0}{\lambda_1}}\,\phi^{-1}(\alpha) + \frac{\sqrt{n}\,(\lambda_0 - \lambda_1)}{\sqrt{\lambda_1}}\right]}$$