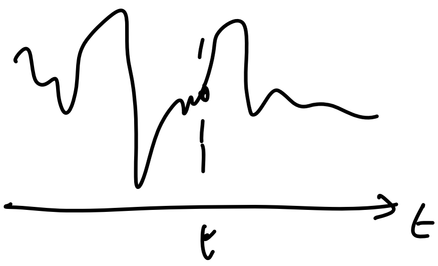
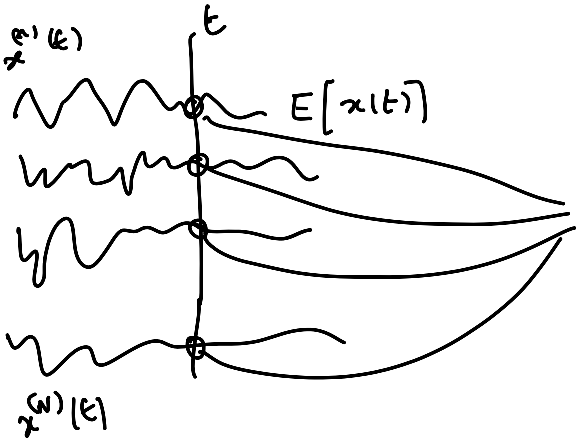
Cours du 5/10



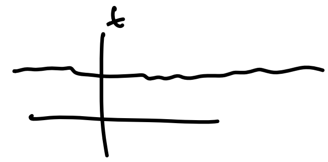
X variable aléatoire

$$E[g(X)] \rightarrow \sum_i g(x_i)\, P[X = x_i]$$

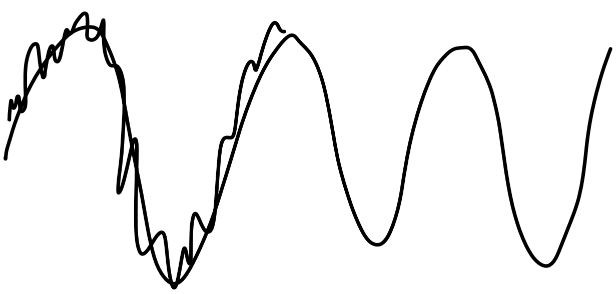
$$E[g(X)] \rightarrow \int g(x)\, \underbrace{p(x)}_{\text{densité de probabilité de } X}\, dx$$



$$\frac{1}{N}\sum_{i=1}^{N} x^{(i)}(t) \xrightarrow[N\to+\infty]{} E[x(t)]$$



Variance d'un signal $\quad E[x^2(t)] - E^2[x(t)]$

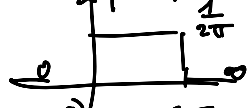


Exemple 1

$x(t) = A\cos(2\pi f_0 t + \theta)$ $\qquad f_0 = 50\,\text{Hz}$

$A = 220\sqrt{2}$

Θ variable aléatoire uniforme sur $]0, 2\pi[$



$$p(\theta) = \begin{cases} \frac{1}{2\pi} & \theta \in]0, 2\pi[\\ 0 & \text{ailleurs} \end{cases}$$

$x(t)$ stationnaire ?

Moyenne $E[x(t)] = E[A\cos(2\pi f_0 t+\theta)] = A\, E[\underbrace{\cos(2\pi f_0 t+\theta)}_{g(\theta)}]$

$$= A \int g(\theta)\, p(\theta)\, d\theta$$

$$= A \int_0^{2\pi} \cos(2\pi f_0 t+\theta)\, \frac{1}{2\pi}\, d\theta$$

$$= \frac{A}{2\pi}\left[\sin(2\pi f_0 t+\theta)\right]_0^{2\pi}$$

$$= 0 = m$$

$$\boxed{\begin{array}{c} E(x(t)) = 0 \\ \text{indépendant de } t \end{array}}$$

$$x^*(t-\tau) = x(t-\tau)$$

$$E(x(t)\, x^*(t-\tau)) = E(A\cos(2\pi f_0 t+\theta)\, \overbrace{A\cos[2\pi f_0 (t-\tau)+\theta]}^{x^*(t-\tau) = x(t-\tau)})$$

$$\cos a \cos b = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

$$= A^2\, E\left[\frac{1}{2}\cos(2\theta + 4\pi f_0 t - 2\pi f_0 \tau) + \frac{1}{2}\cos(2\pi f_0 \tau)\right]$$

$$= \frac{A^2}{2}\left[\int_0^{2\pi} \frac{\cos(2\theta+4\pi f_0 t - 2\pi f_0\tau)}{2\pi}\, d\theta + \int_0^{2\pi} \frac{\cos(2\pi f_0 \tau)}{2\pi}\, d\theta\right]$$

$$= \frac{A^2}{2}(0 + \cos(2\pi f_0 \tau))$$

$$\boxed{E[x(t)\, x^*(t-\tau)] = \frac{A^2}{2}\cos(2\pi f_0 \tau)}$$

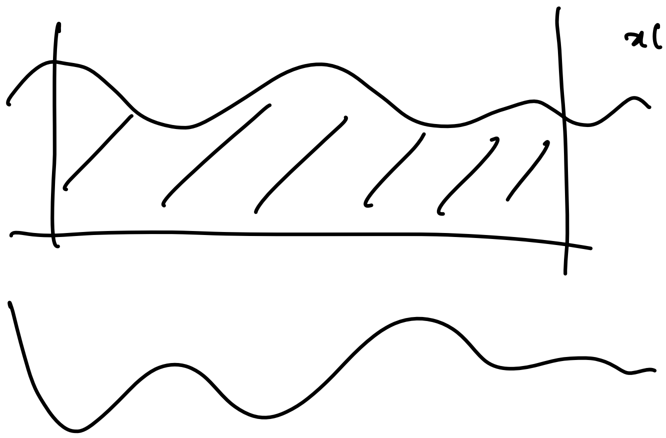
ind de t

$E[x(t)]$ ind de t
$E(x(t)x^*(t-\tau))$ ind de t $\Rightarrow$ $\boxed{x(t) \text{ stationnaire}}$

Transformée de Fourier

$$\int_{\mathbb{R}} x(t)\, e^{-j2\pi f t}\, dt$$

$x(t)e^{-j2\pi f t}$



Définition : densité spectrale de puissance d'un signal aléatoire

$$S_x(f) = TF[R_x(\tau)]$$

$$= \int_{\mathbb{R}} R_x(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$\cancel{TF[x(t)]}$

définit le spectre des signaux aléatoires

$$R_x(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau) \overset{\text{Table}}{\Longrightarrow} S_x(f) = TF[R_x(\tau)]$$

$$= \boxed{\frac{A^2}{4}\left(\delta(f-f_0) + \delta(f+f_0)\right)}$$

$$S_x(f) = TF[R_x(\tau)] \Rightarrow R_x(\tau) = TF^{-1}(S_x(f))$$

$$= \int R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad\qquad = \int S_x(f)\, e^{+j2\pi f\tau}\, df$$

$$\boxed{P = R_x(0) = \int S_x(f)\, df}$$

$$d^2[x(t), x(t-\tau)] = \| x(t) - x(t-\tau) \|^2$$

$$= \langle x(t) - x(t-\tau),\ x(t) - x(t-\tau) \rangle$$

$$= \langle x(t), x(t) \rangle + \langle x(t-\tau), x(t-\tau) \rangle$$

$$- \underbrace{\langle x(t), x(t-\tau) \rangle}_{E[x(t)x^*(t-\tau)] = R_x(\tau)} - \langle x(t-\tau), x(t) \rangle$$

$E[x(t)x^*(t)] = E[|x(t)|^2]$

$R_x(-\tau)$

$E[x(t-\tau)x^*(t-\tau)] = R_x(0)$

Remarque

$E(x(t_1)x^*(t_2)) = R_x(t_1 - t_2)$

$\tau = t - (t-\tau)$

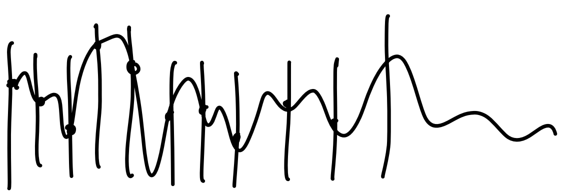
$$d^2[x(t), x(t-\tau)] = R_x(0) + R_x(0) - R_x(\tau) - \underbrace{R_x(-\tau)}_{R_x^*(\tau)}$$

<u>Si $x(t)$ réel</u>

$$\boxed{d^2[x(t), x(t-\tau)] = 2\,[R_x(0) - R_x(\tau)]}$$

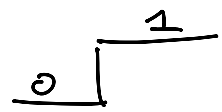
Ergodicité ?

$E(x(t)) = ?$



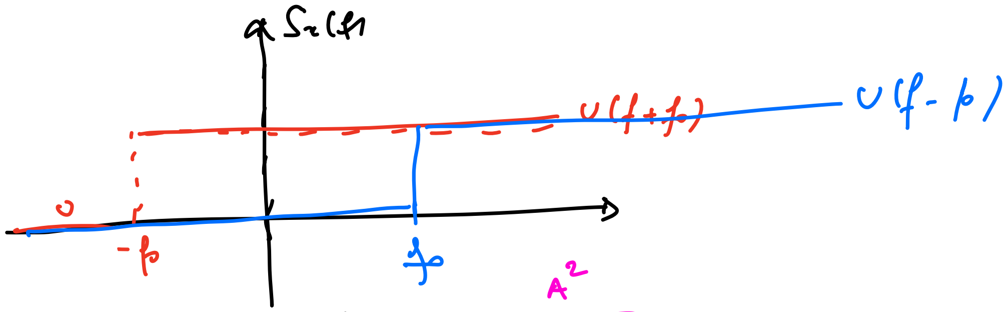
t

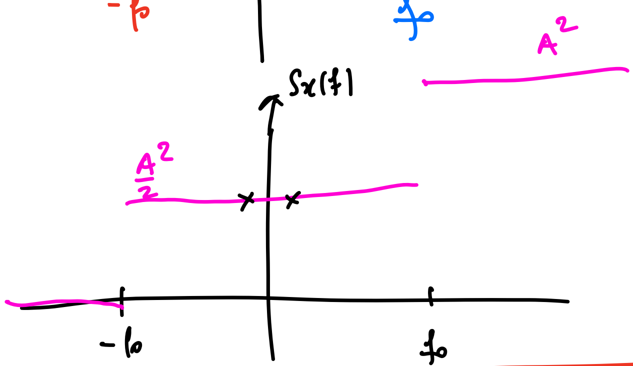
$(U(f))' = \delta(f)$



$$\Delta_x(f) = \frac{A^2}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right]$$

$$S_x(f) = \frac{A^2}{2}\left[U(f-f_0)+U(f+f_0)\right]$$





$$\Delta S(0) = \frac{A^2}{2} - \frac{A^2}{2} = \boxed{0}$$

<u>Exemple de signal non ergodique</u>

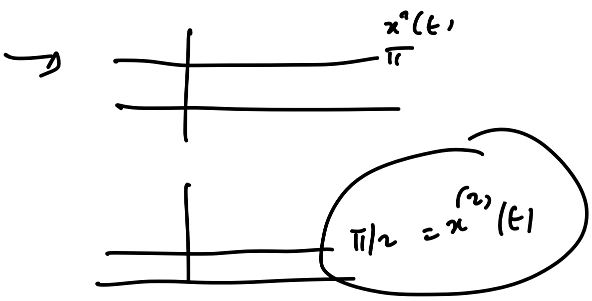
$x(t) = \theta$ $\quad$ θ variable aléatoire uniforme sur $]0, 2\pi[$.

$$E(x(t)) = E(\theta) = \pi$$

$$\int_0^{2\pi} \frac{\theta}{2\pi}\, d\theta = \frac{1}{2\pi}\left[\frac{\theta^2}{2}\right]_0^{2\pi} = \pi$$

$$E\left(x(t)x(t-\tau)\right) = E(\theta^2) = \frac{(2\pi)^2}{12} = \frac{\pi^2}{3}$$

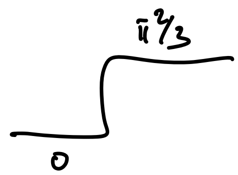
$$\int_0^{2\pi} \frac{\theta^2}{2\pi}\, d\theta$$



$$R_x(\tau) = \frac{\pi^2}{3}$$

$$S_x(f) = \frac{\pi^2}{3}\,\delta(f)$$

$$S_x(f) = \frac{\pi^2}{3}\,U(f)$$



$$\Delta S_x(0) = \frac{\pi^2}{3} \neq E^2(x(t)) = \pi^2$$