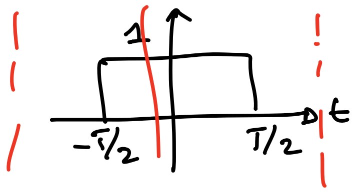
Cours du 22/01/2024

$x(t) = \Pi_{T}(t)$ largeur de la fenêtre



Fonction d'autocorrélation $R_x(\tau)$?

Densité spectrale ? $S_x(f)$

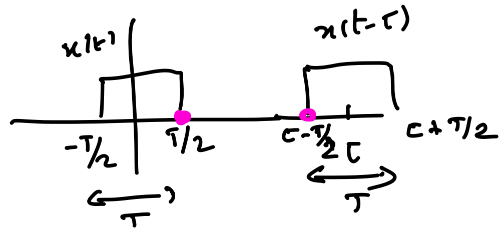
$$x(t) * y(t) = \int x(u)\, y(t-u)\, du$$

Méthode 1

1) Calcul de $R_x(\tau)$

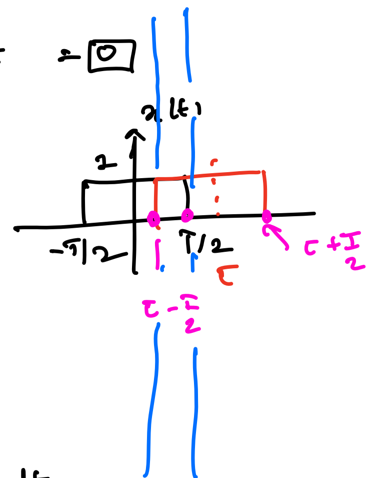
$$R_x(\tau) = \int_{\mathbb{R}} x(t)\, x^*(t-\tau)\, dt$$

conjugué



1er cas. $\tau - \frac{T}{2} > \frac{T}{2} \iff \boxed{\tau > T}$

$$R_x(\tau) = \int 0 \, dt = \boxed{0}$$

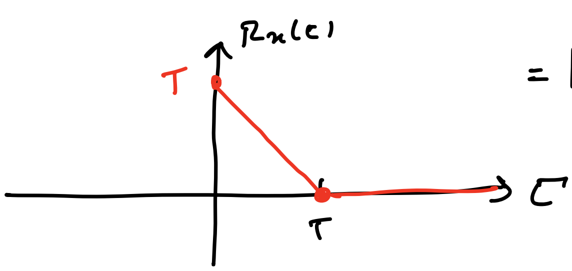


2ème cas. $\begin{cases} \tau - \frac{T}{2} < \frac{T}{2} \\ \tau + \frac{T}{2} > \frac{T}{2} \end{cases}$

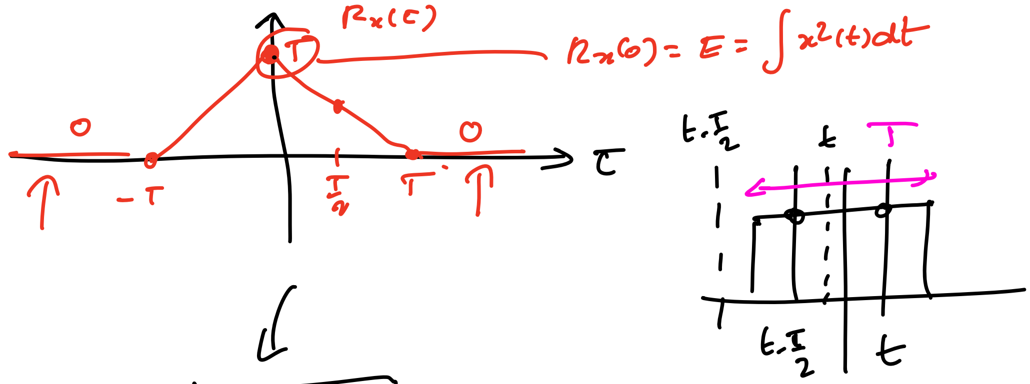
$\Updownarrow$ $\boxed{\tau \in]0, T[}$

$$R_x(\tau) = \int_{\tau - \frac{T}{2}}^{T/2} 1 \times 1 \, dt$$

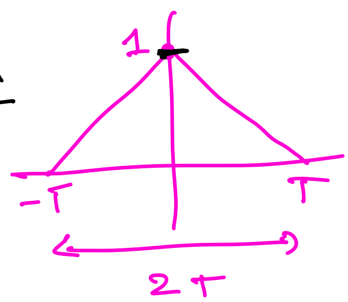
$$= \boxed{T - \tau}$$



On montrera que $R_x(\tau)$ est une fonction paire pour un signal réel donc on en déduit $R_x(\tau)$ pour $\tau < 0$



$$\boxed{\Lambda_T(\tau)}$$

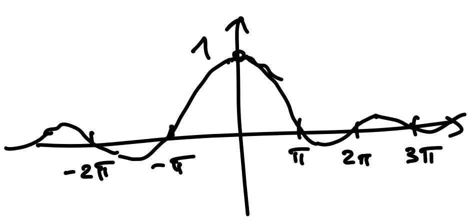


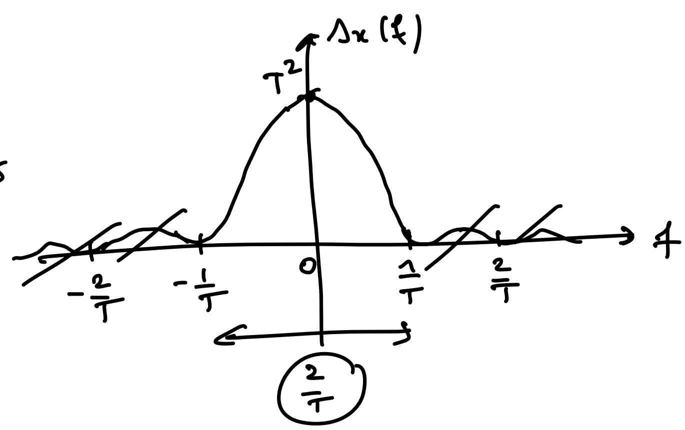
donc $\boxed{R_x(\tau) = T \Lambda_T(\tau)}$

Densité spectrale du signal $x(t)$

$$S_x(f) = TF[R_x(\tau)] = T \times T \operatorname{sinc}^2(\pi T f)$$
$$= \boxed{T^2 \operatorname{sinc}^2(\pi T f)}$$

$\operatorname{sinc}(x) = \frac{\sin x}{x}$





2) <u>Méthode 2</u>

- <u>Calcul de</u> $S_x(f) = |X(f)|^2$

$$x(t) = \Pi_T(t) \xrightarrow{TF} X(f) = T\,\mathrm{sinc}(\pi T f)$$

$\downarrow$ Densité spectrale

$$S_x(f) = |X(f)|^2 = \boxed{T^2\,\mathrm{sinc}^2(\pi T f)}$$

- <u>Autocorrélation</u>

$$R_x(\tau) = TF^{-1}\left(S_x(f)\right) = T\;TF^{-1}\left(T\,\mathrm{sinc}^2(\pi T f)\right)$$
$$= \boxed{T\,\Lambda_T(\tau)}$$

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,x^*(t-\tau)\,dt$$

$x(t) = \sum_k c_k e^{j2\pi k \frac{t}{T_0}}$; $x^*(t-\tau) = \sum_\ell c_\ell^* e^{-j2\pi \ell \frac{t-\tau}{T_0}}$

<u>Exo 2</u>

$$\boxed{\begin{array}{c} x(t) = A\cos(2\pi f_0 t) \\ R_x(\tau)\,?\quad S_x(f)\,? \end{array}}$$

<u>Méthode 1</u>

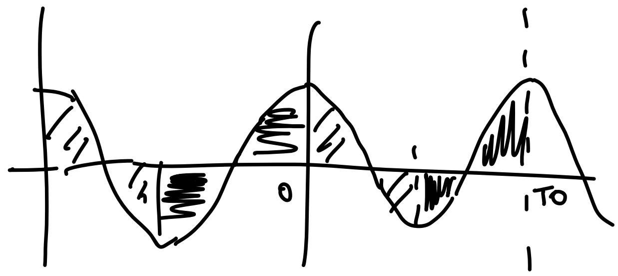
- <u>fonction d'autocorrélation</u>

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A\cos(2\pi f_0 t)\,A\cos[2\pi f_0(t-\tau)]\,dt$$

$$\cos a\cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right]$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \underbrace{\frac{A^2}{2}}\left[\cos(4\pi f_0 t - 2\pi f_0\tau) + \cos(2\pi f_0\tau)\right]dt$$

$$= 0 + \left(\frac{1}{T_0}\int^{T_0/2} dt\right)\cos(2\pi f_0\tau)\,\frac{A^2}{2}$$

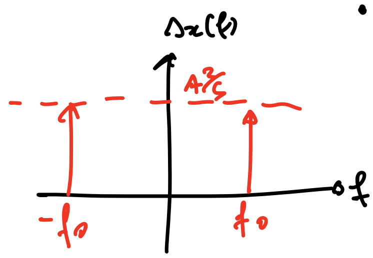


$-T_0/2$

$$= \boxed{\frac{A^2}{2}\cos(2\pi f_0 \tau)}$$

C'est une fonction périodique de période T_0 !!

- **Densité spectrale**



$$S_x(f) = TF[\,R_x(\tau)\,]$$

$$= \boxed{\frac{A^2}{4}\left[\delta(f-f_0) + \delta(f+f_0)\right]}$$

2 fréquences pures !

$\boxed{\text{Méthode 2}}$

- **densité spectrale**

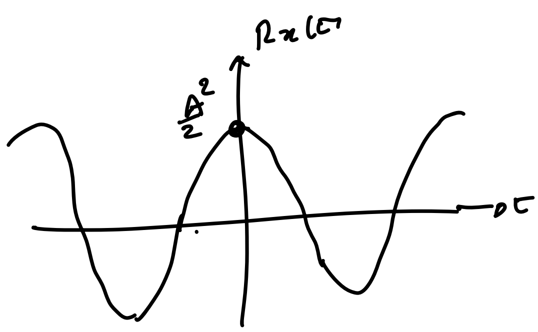
$$x(t) = A\cos(2\pi f_0 t) = \underbrace{\frac{A}{2}}_{c_1} e^{j2\pi f_0 t} + \underbrace{\frac{A}{2}}_{c_{-1}} e^{-j2\pi f_0 t}$$

$$\boxed{\sum_k c_k\, e^{j2\pi k f_0 t}}$$

$$\boxed{S_x(f) = \frac{A^2}{4}\delta(f-f_0) + \frac{A^2}{4}\delta(f+f_0)}$$

- **Autocorrélation**

$$R_x(\tau) = \frac{A^2}{4}\underbrace{TF^{-1}\left[\delta(f-f_0)\right]}_{e^{j2\pi f_0 \tau}} + \frac{A^2}{4}\underbrace{TF^{-1}\left[\delta(f+f_0)\right]}_{e^{-j2\pi f_0 \tau}}$$



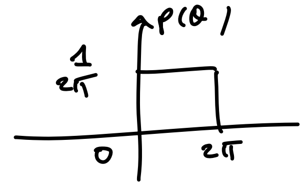
$$= \boxed{\frac{A^2}{2}\cos(2\pi f_0 \tau)}$$

Rq : $\tau = 0$ $\boxed{R_x(0) = \text{Puissance} = \frac{A^2}{2}}$

Cours du 24/01/2024

$\boxed{\text{Exemple}}$

$x(t) = A \cos(2\pi f_0 t + \theta)$ θ uniforme sur $]0, 2\pi[$



$R_x(\tau)$? $S_x(f)$?

$$R_x(\tau) = E\left[x(t)\, x^*(t-\tau) \right]$$

$$= E\Big[\underbrace{A \cos(2\pi f_0 t + \theta)\; A \cos(2\pi f_0 (t-\tau) + \theta)}_{h(\theta)} \Big]$$

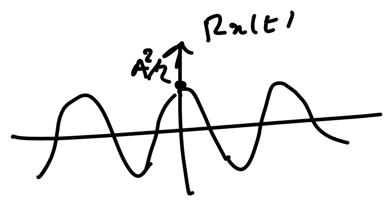
$$= \int h(\theta)\, p(\theta)\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \frac{A^2}{2} \left[\cos(2\theta + \dots) + \cos(2\pi f_0 \tau) \right] d\theta$$

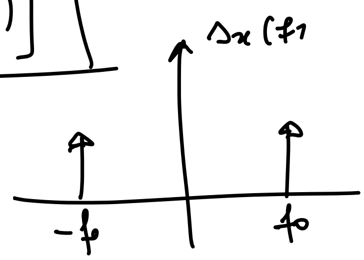
$$\boxed{\cos a \cos b = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)}$$

$$= \frac{A^2}{4\pi} \left[0 + \cos(2\pi f_0 \tau)\; 2\pi \right]$$

$$= \boxed{\frac{A^2}{2} \cos(2\pi f_0 \tau)}$$



$$S_x(f) = \boxed{\frac{A^2}{4} \left[\delta(f - f_0) + \delta(f + f_0) \right]}$$



$$S_x(f) = TF[R_x(\tau)] = \int R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$R_x(\tau) = TF^{-1}[S_x(f)] = \int S_x(f)\, e^{j2\pi f\tau}\, df$$

$\boxed{R_x(0) = \int S_x(f)\, df \text{ est la puissance du signal } x(t)}$

$$d^2[x(t), x(t-\tau)] = 2[R_x(0) - R_x(\tau)]$$

$$\|x(t) - x(t-\tau)\|^2 = \langle x(t) - x(t-\tau), x(t) - x(t-\tau)\rangle$$

<u>Cas des signaux aléatoires</u>

$$E\left[(x(t) - x(t-\tau))(x(t) - x(t-\tau))^*\right]$$

$R_x(-\tau) = R_x(\tau)$ pour un signal réel

$$E[x(t)x^*(t)] - E[x(t)x^*(t-\tau)] - \underbrace{E[x(t-\tau)x^*(t)]}_{} + E[x(t-\tau)x^*(t-\tau)]$$

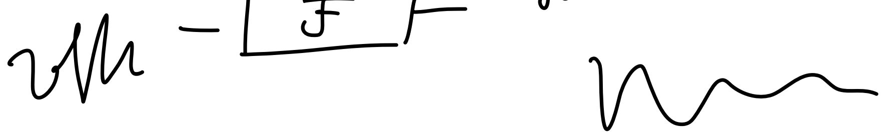
$$R_x(0) - R_x(\tau) \qquad R_x(0)$$

$$\boxed{E[x(t_1)x^*(t_2)] \overset{?}{=} R_x(t_1 - t_2)}$$

$$d^2[x(t), x(t-\tau)] = 2[R_x(0) - R_x(\tau)] \quad \underline{\underline{\text{ok}}}$$

FILTRAGE

$x(t)$ — $\boxed{\mathcal{F}}$ — $y(t) = x(t) * h(t)$



$$= \int x(u)\, h(t-u)\, du$$

$$y(t) = \int h(u)\, x(t-u)\, du$$

$a(t) = x(t-t_0)$ — $\boxed{\mathcal{F}}$ — ?

$$\int h(u)\, a(t-u)\, du = \int h(u)\, x(t-t_0-u)\, du$$

$$= y(t-t_0)$$

<u>Exemple de transformation non invariante dans le temps</u>

$x(t)$ — $\boxed{\otimes}$ — $y(t) = x(t)\, m(t)$

$m(t)$

$\boxed{\text{opération de modulation}}$

C'est une opération linéaire

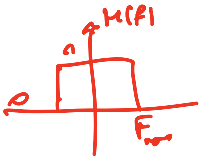
$x(t-t_0)$ — $\boxed{}$ — $x(t-t_0)\, m(t) \neq y(t-t_0) = x(t-t_0)\, m(t-t_0)$

$h(t)$ s'appelle la <u>réponse impulsionnelle</u> du filtre

$x(t)$ — $\boxed{\mathcal{F}}$ → $y(t) = x(t) * h(t)$

$x(t) = \delta(t)$ — $\boxed{}$ — $\delta(t) * h(t) = \boxed{h(t)}$

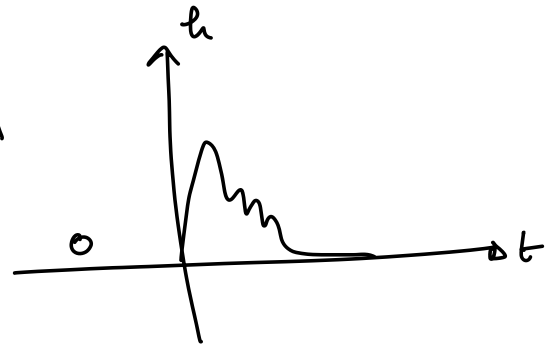
<u>Rappel</u> $\boxed{\delta(t-t_0) * h(t) = h(t-t_0)}$



$$\boxed{S_y(f) = S_x(f)\,|H(f)|^2}$$

$H(f) = TF[\,h(t)\,]$ s'appelle la transmittance du filtre

$$\boxed{h \text{ causale si } h(t) = 0 \quad t < 0}$$



$$y(t) = \int_{\mathbb{R}} h(u)\,x(t-u)\,du = \int_0^{+\infty} h(u)\,x(t-u)\,du$$