Cours du 15/01/2024

$X_i \sim \mathcal{N}(m, \sigma^2)$ σ^2 connue $\theta = m$

m : $E[X_i]$; σ^2 : $Var(X_i)$

$$\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\tilde{\theta}_n = \frac{2}{n(n+1)} \sum_{i=1}^{n} (i X_i)$$

Lequel des deux estimateurs est le meilleur ?

Biais.

$$\text{Biais de } \hat{\theta}_n = E[\hat{\theta}_n] - \underbrace{\theta}_{=m}$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] - m$$

$$= \frac{1}{n}\sum_{i=1}^{n} \underbrace{E[X_i]}_{m} - m$$

linéarité de E

$$= \frac{1}{n}\underbrace{\sum_{i=1}^{n} m}_{nm} - m = \boxed{0}$$

l'estimateur $\hat{\theta}_n$ est non-biaisé (biais égal à 0)

$$\text{Biais de } \tilde{\theta}_n : E[\tilde{\theta}_n] - \theta = E\left[\frac{2}{n(n+1)}\sum_{i=1}^{n} iX_i\right] - m$$

$$= \frac{2}{n(n+1)} \sum_{i=1}^{n} \underbrace{E[iX_i]}_{iE[X_i]} - m$$

$$= \frac{2m}{n(n+1)} \underbrace{\left(\sum_{i=1}^{n} i\right)}_{\frac{n(n+1)}{2}} - m$$

$E(X_i) = m$

$$= m - m = \boxed{0}$$

$\tilde{\theta}_n$ est aussi un estimateur non biaisé de $\theta = m$

Variance des estimateurs

$$\boxed{Var(aX + b) = a^2 \, Var X}$$

Si X et Y sont ind alors $Var(X+Y) = Var X + Var Y$

$$\mathrm{Var}[\hat\theta_n] = \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^n X_i\right]$$

$$= \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^n X_i\right)$$

$X_1, \dots, X_n$ iid

$$= \frac{1}{n^2}\sum_{i=1}^n \underbrace{\mathrm{Var}\,X_i}_{\sigma^2} = \frac{n\sigma^2}{n^2} = \boxed{\frac{\sigma^2}{n}}$$

$$\mathrm{Var}(\tilde\theta_n) = \mathrm{Var}\left[\frac{2}{n(n+1)}\sum_{i=1}^n iX_i\right]$$

les va iX_i sont ind

$$= \frac{4}{n^2(n+1)^2}\sum_{i=1}^n \underbrace{\mathrm{Var}(iX_i)}_{i^2\,\mathrm{Var}X_i} = i^2\sigma^2$$

$$= \frac{4\sigma^2}{n^{\cancel{2}}(n+1)^{\cancel{2}}}\underbrace{\sum_{i=1}^n i^2}_{\frac{\cancel{n}\cancel{(n+1)}(2n+1)}{6}} \quad (\text{Admis})$$

$$= \frac{2}{3}\sigma^2\frac{2n+1}{n(n+1)} = \frac{\sigma^2}{n}\times\left[\frac{2}{3}\frac{2n+1}{n+1}\right]$$

$$\underset{n\to+\infty}{\approx} \frac{4}{3} > 1$$

donc $\mathrm{Var}[\tilde\theta_n] > \mathrm{Var}[\hat\theta_n]$ pour n "grand"

donc on préfèrera $\hat\theta_n$

$$\boxed{E[(\hat\theta-\theta)^2] = (\text{biais})^2 + \text{Variance}}$$

$$E\left[\left(\hat\theta - E(\hat\theta) + E(\hat\theta) - \theta\right)^2\right] = \underbrace{E\left[\left(\hat\theta - E(\hat\theta)\right)^2\right]}_{\mathrm{Var}\,\hat\theta}$$

$$+ E\left[\underbrace{(E(\hat\theta)-\theta)^2}_{(\text{biais})^2}\right] + 2E\left[\underbrace{(E(\hat\theta)-\theta)}_{\text{non aléa...}}\left(\hat\theta - E(\hat\theta)\right)\right]$$

va aléatoire

donc $E\left((\hat\theta-\theta)^2\right) = \operatorname{Var}\hat\theta + (\text{biais})^2 + 2\left(E(\hat\theta)-\theta\right) E\left[\hat\theta - E(\hat\theta)\right]$

$E[\hat\theta] - E[E(\hat\theta)] = 0$, avec $E[E(\hat\theta)] = E(\hat\theta)$

donc $\boxed{\text{MSE ou EQM} = \text{variance} + (\text{biais})^2}$

MSE = mean square error

EQM = erreur quadratique moyenne

$$E[\otimes] \longrightarrow \frac{1}{n}\sum_{i=1}^{n} \otimes$$

Exemple 2

$X_i \sim N(m, \sigma^2)$ $\quad \theta = \sigma^2$

$\hat\sigma_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar X_n)^2 \qquad \bar X_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

l'estimateur $\hat\sigma_n^2$ est-il biaisé ou non ?

$$E[\hat\sigma_n^2] = \frac{1}{n}\sum_{i=1}^{n} \underbrace{E\left[(X_i - m + m - \bar X_n)^2\right]}_{E[(X_i-m)^2] + E[(m-\bar X_n)^2] + 2E[(X_i-m)(m-\bar X_n)]}$$

σ^2 : $E[(X_i-m)^2]$

variance de $\bar X_n$ grand $\bar X_n = m$: $E[(m-\bar X_n)^2] = \frac{\sigma^2}{n}$ calculé avant

$$-2E\left[(X_i-m)\left(\frac{1}{n}\sum_{k=1}^{n} X_k - m\right)\right] = -2E\left[\frac{(X_i-m)}{n}\sum_{k=1}^{n}(X_k - m)\right]$$

$m = \frac{1}{n}\sum_{k=1}^{n} m$

$$= -\frac{2}{n}\sum_{k=1}^{n} E\left[\underbrace{(X_i-m)(X_k-m)}_{cov(X_i,X_k) = \begin{cases} 0 & i\neq k \\ \sigma^2 & i=k\end{cases}}\right]$$

$$= \boxed{-\frac{2}{n}\sigma^2}$$

$\frac{n}{n-1}\hat{\sigma}_n^2$

$$\left(= \frac{1}{n}\sum_{k=1}^{n} a = a\right)$$

Conclusion

$$E\left[\hat{\sigma}_n^2\right] = \sigma^2 + \frac{\sigma^2}{n} - \frac{2\sigma^2}{n} = \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2 \neq \sigma^2$$

$\times \frac{n}{n-1}$

donc $\hat{\sigma}_n^2$ est un estimateur biaisé de σ^2

$$\boxed{\text{Un bon estimateur de } \sigma^2 \text{ est } (\sigma^2)^* = \frac{n}{n-1}\hat{\sigma}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}$$

vraisemblance

$X_i \sim N(m, \sigma^2)$
Loi normale

$$L(x_1 \dots x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i-m)^2}{2\sigma^2}\right]$$

$X_i \sim P(\lambda)$

Loi de Poisson

$$L(x_1 \dots x_n) = \prod_{i=1}^{n}\left(\frac{\lambda^{x_i}}{x_i!}e^{-\lambda}\right)$$

Borne de Cramer-Rao pour un estimateur non biaisé de $\theta = m$
quand $X_i \sim N(m, \sigma^2)$

log vraisemblance $\sum_{i=1}^{n}\left[-\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x_i-m)^2\right]$

$$\frac{\partial \ln L}{\partial m} = \sum_{i=1}^{n} -\frac{1}{2\sigma^2}\,2(x_i-m)(-1) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-m)$$

$$\frac{\partial^2 \ln L}{\partial m^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(-1) = -\frac{n}{\sigma^2}$$

$$E\left[\frac{\partial^2 \ln L}{\partial m^2}\right] = E\left[-\frac{n}{\sigma^2}\right] = -\frac{n}{\sigma^2} \Rightarrow \boxed{BCR = \frac{\sigma^2}{n}}$$

On connaît un estimateur non biaisé de m qui est $\hat{m} = \frac{1}{n}\sum_{i=1}^{n} x_i$

dont la variance est égale à $\frac{\sigma^2}{n}$

C'est le meilleur estimateur possible de m

$\boxed{\text{Cours du } 5/02/2024}$

$X_1, \dots, X_n$ échantillon de loi $\Gamma(v, \theta)$
Estimateur des moments de $\binom{v}{\theta}$?

TABLES

$$\begin{cases} E[X_i] = \dfrac{v}{\theta} \\ E[X_i^2] = \operatorname{Var} X_i + E^2[X_i] = \dfrac{v}{\theta^2} + \dfrac{v^2}{\theta^2} \end{cases}$$

Inversion de ce système

$$\begin{cases} v = \theta\, E[X_i] \\ E[X_i^2] = \dfrac{E[X_i]}{\theta} + E^2[X_i] \end{cases} \iff \begin{cases} \left(E[X_i^2] - E^2[X_i]\right)\theta = E[X_i] \end{cases}$$

donc $\boxed{\theta = \dfrac{E[X_i]}{E[X_i^2] - E^2[X_i]}}$

$$\boxed{v = \frac{E^2[X_i]}{E[X_i^2] - E^2[X_i]}}$$

Estimateurs des moments

$$\boxed{\begin{aligned} \hat{\theta}_{Mo} &= \frac{\frac{1}{n}\sum_{i=1}^n x_i}{\frac{1}{n}\sum_{i=1}^n x_i^2 - \left(\frac{1}{n}\sum x_i\right)^2} \\ \hat{v}_{Mo} &= \frac{\left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2}{\frac{1}{n}\sum_{i=1}^n x_i^2 - \left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2} \end{aligned}}$$

BAYES

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

$$\underbrace{p(\theta | x_1, \dots, x_n)}_{\boxed{\text{Loi a posteriori de } \theta}} = \frac{\overbrace{p(x_1, \dots, x_n | \theta)}^{\text{vraisemblance}}\ \overbrace{p(\theta)}^{\text{loi a priori}}}{\underbrace{p(x_1, \dots, x_n)}_{\text{"constante" de normalisation}}}$$

$\underset{\text{proportionnel à}}{\propto} \quad p(x_1, \dots, x_n | \theta)\ p(\theta)$

MMSE = minimum mean square error.

$\hat{\theta}_{MMSE}$ = moyenne de la loi a posteriori qui minimise $E[(\hat{\theta}-\theta)^2]$

$= E[\theta \mid x_1, \dots, x_n]$

Exemple

$x_i \sim N(\theta, \sigma^2) \Rightarrow$ vraisemblance

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i-\theta)^2}{2\sigma^2}\right] \quad (*)$$

Loi apriori $\quad \theta \sim N(\mu, v^2) \quad \mu$ et v connus

$$p(\theta) = \frac{1}{\sqrt{2\pi v^2}} \exp\left[-\frac{(\theta-\mu)^2}{2v^2}\right] \quad (*)$$

Estimateurs MAP et MMSE de θ ?

Loi aposteriori

$$p(\theta / x_1, \dots x_n) \propto \boxed{\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2\right] \times \exp\left[-\frac{(\theta-\mu)^2}{2v^2}\right]}$$

$$\ln p(\theta / x_1, \dots, x_n) = K - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2 - \frac{1}{2v^2}(\theta-\mu)^2$$

$$\frac{\partial \ln p}{\partial \theta} = 0 \iff -\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n} \cancel{2}(x_i-\theta)(-1) - \frac{1}{\cancel{2}v^2}\cancel{2}(\theta-\mu) = 0$$

$$\iff \underbrace{\frac{\theta}{v^2} + \frac{n\theta}{\sigma^2}} = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i + \frac{\mu}{v^2}$$

$$\theta\left(\frac{1}{v^2} + \frac{n}{\sigma^2}\right)$$

$$\theta\left(\frac{\sigma^2 + nv^2}{v^2\sigma^2}\right)$$

d'où

$$\boxed{\theta = \frac{v^2}{\sigma^2+nv^2}\sum_{i=1}^{n} x_i + \frac{\sigma^2}{\sigma^2+nv^2}\mu}$$

α_n

donc

$$\boxed{\hat{\theta}_{MAP} = \frac{nv^2}{nv^2+\sigma^2}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)} + \frac{\sigma^2}{\sigma^2+nv^2}\mu}$$

α_n $\quad \underbrace{\hat{\theta}_{MV}}_{\text{vient des données}}$ $\quad$ vient de la loi a priori

n "grand" $n \to +\infty$	$\alpha_n \to 1$ $1-\alpha_n \to 0$	$\hat{\theta}_{MAP} \approx \hat{\theta}_{MV}$ On fait confiance aux données !
n "petit" $n \to 0$	$\alpha_n \to 0$ $1-\alpha_n \to 1$	$\hat{\theta}_{MAP} \approx \mu$ on fait confiance à l'information a priori !
$\sigma^2 \to 0$	$\alpha_n \approx 1$ $1-\alpha_n \approx 0$	on fait confiance aux données !
$\sigma^2 \to +\infty$	$\alpha_n \approx 0$ $1-\alpha_n \approx 1$	on fait confiance à l'info a priori !

Estimateur MMSE

$$\hat{\theta}_{MMSE} = E[\theta \mid x_1, \dots, x_n]$$

$\boxed{\text{TD du 5/02}}$

$$f(x) = \begin{cases} \frac{\alpha}{\theta} x^{\alpha-1} \exp\left(-\frac{x^\alpha}{\theta}\right) & x>0 \\ 0 & x \le 0 \end{cases}$$

1) Loi de $U = X^\alpha$ $\iff x = U^{1/\alpha}$

$$g(u) = \frac{\alpha}{\theta} \, U^{\frac{\alpha-1}{\alpha}} \exp\left(-\frac{u}{\theta}\right) \Big| \underbrace{\frac{dx}{du}}_{\frac{1}{\alpha} u^{1/\alpha - 1}} \Big|$$

$$\boxed{g(u) = \frac{1}{\theta} \exp\left(-\frac{u}{\theta}\right)}$$

Domaine de U $\quad x>0 \iff u^{1/\alpha} > 0 \iff u>0$

donc $\boxed{g(u) = \frac{1}{\theta} \exp\left(-\frac{u}{\theta}\right) \mathbb{1}_{\mathbb{R}^+}(u)}$

$$U \sim \Gamma\Big(\underbrace{1}_{a}, \underbrace{\tfrac{1}{\theta}}_{b}\Big)$$

$$\boxed{\begin{aligned} E[U] &= \frac{a}{b} = \theta \\ Var(U) &= \frac{a}{b^2} = \theta^2 \end{aligned}}$$

2) Estimateur du max de vraisemblance de θ

vraisemblance $\quad L(x_1,\dots,x_n;\theta) = \prod_{i=1}^{n}\left[\frac{\lambda}{\theta}\, x_i^{\lambda-1} \exp\left(-\frac{x_i^{\lambda}}{\theta}\right)\right]$

Log vraisemblance $\quad \ln L = \sum_{i=1}^{n}\left[\ln\lambda - \ln\theta + \ln\left(x_i^{\lambda-1}\right) - \frac{x_i^{\lambda}}{\theta}\right]$

max de vraisemblance $\quad \frac{\partial \ln L}{\partial \theta} \overset{\geq 0}{=} 0 \iff -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i^{\lambda} \overset{\geq 0}{=} 0$

$$\iff \boxed{\theta \overset{\leq}{=} \frac{1}{n}\sum_{i=1}^{n} x_i^{\lambda}}$$

donc $\quad \boxed{\hat{\theta}_{MV} = \frac{1}{n}\sum_{i=1}^{n} X_i^{\lambda}}$

θ	0		$\frac{1}{n}\sum_{i=1}^{n} x_i^{\lambda}$		$+\infty$
$\frac{\partial \ln L}{\partial \theta}$		$+$	0	$-$	
$\ln L$		$\nearrow$		$\searrow$	

Estimateur sans biais ? convergent ? efficace ?

<u>Biais</u> $\quad b_n(\theta) = E[\hat{\theta}] - \theta$

$$E(\hat{\theta}) = E\left[\frac{1}{n}\sum X_i^{\lambda}\right] = \frac{1}{n}\sum_{i=1}^{n} E[\underbrace{X_i^{\lambda}}_{U_i}]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \theta = \boxed{\theta}$$

donc $\hat{\theta}$ non biaisé

Variance $\quad Var[\hat{\theta}] = Var\left[\frac{1}{n}\sum_{i=1}^{n} x_i^{d}\right]$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(\underbrace{x_i^{d}}_{v_i}) = \frac{1}{n^2}\sum_{i=1}^{n} \theta^2$$

$$= \boxed{\frac{\theta^2}{n}}$$

$x_1, \dots, x_n$ ind

$\Rightarrow x_1^{d}, \dots, x_n^{d}$ ind

$$\boxed{\left.\begin{array}{l} Biais(\hat{\theta}) = 0 \\ Var\,\hat{\theta} \xrightarrow[n \to +\infty]{} 0 \end{array}\right. \Rightarrow \hat{\theta} \text{ convergent}}$$

Efficacité $\quad BCR = \dfrac{1}{E\left[-\dfrac{\partial^2 \ln L}{\partial \theta^2}\right]}$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i^{d}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i^{d}$$

$$E[\ ] = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} \underbrace{E(x_i^{d})}_{\theta} = \frac{n}{\theta^2} - \frac{2n}{\theta^2} = -\frac{n}{\theta^2}$$

donc $\boxed{BCR = \frac{\theta^2}{n}}$

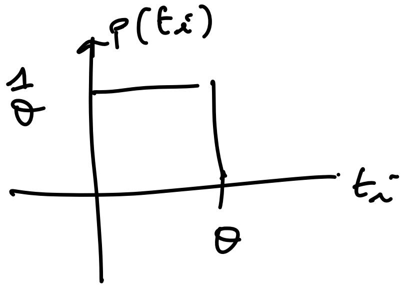
$$\left\{\begin{array}{l} \hat{\theta} \text{ non biaisé} \\ Var\,\hat{\theta} = BCR \end{array}\right. \Rightarrow \boxed{\hat{\theta} \text{ est l'estimateur efficace de } \theta}$$

Erreur quadratique moyenne

$$E[(\hat{\theta}-\theta)^2] = \text{Biais}^2 + \text{Variance} = \boxed{\frac{\theta^2}{n}}$$

Exo 1

1)



$$E(T_i) = \frac{\theta}{2}$$

$$\underbrace{Var(T_i) = \frac{\theta^2}{12}}_{\text{Table}}$$

2) $\bar{T} = \frac{1}{n}\sum_{i=1}^{n} T_i \quad (= \hat{\theta})$

$$E(\bar{T}) = \frac{1}{n}\sum_{i=1}^{n} \underbrace{E(T_i)}_{\frac{\theta}{2}} = \frac{\theta}{2}$$

donc $E(\bar{T}) - \theta = \frac{\theta}{2} - \theta = -\frac{\theta}{2} \neq 0$

donc $\bar{T}$ estimateur <u>biaisé</u> de θ

$\hat{\theta} = 2\bar{T} \qquad E(\hat{\theta}) = 2 \times \frac{\theta}{2} = \theta$

<u>$\hat{\theta}$ non biaisé</u>

$$Var\,\hat{\theta} = 4\,Var(\bar{T}) = 4\,Var\left(\frac{1}{n}\sum T_i\right)$$

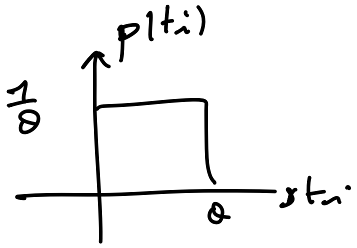
$$= \frac{4}{n^2}\sum_{i=1}^{n} \underbrace{Var\,T_i}_{\frac{\theta^2}{12}}$$

$$= \boxed{\frac{\theta^2}{3n}}$$

$$\begin{cases} \hat{\theta} \text{ non biaisé} \\ \operatorname{Var} \hat{\theta} \xrightarrow[n \to +\infty]{} 0 \end{cases} \quad \text{donc} \quad \boxed{\hat{\theta} \text{ estimateur convergent}}$$

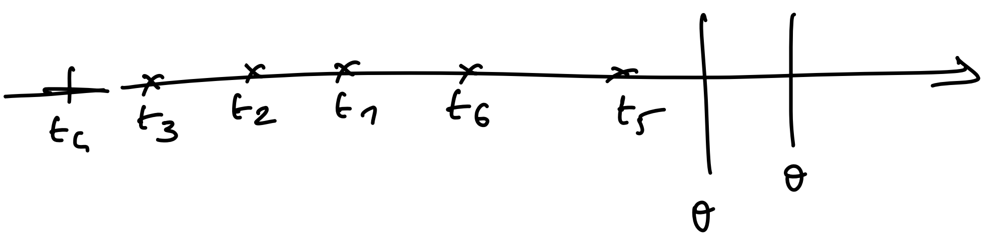
3) <u>Estimateur du max de vraisemblance</u>

$$L(t_1, \dots, t_n ; \theta) = \prod_{i=1}^{n} \left[\frac{1}{\theta} \mathbb{1}_{]0,\theta[}(t_i) \right]$$



$$= \begin{cases} \frac{1}{\theta^n} & \text{si tous les } t_i \in]0, \theta[\\ 0 & \text{sinon} \end{cases}$$

$$p(t_i) = \frac{1}{\theta} \mathbb{1}_{]0,\theta[}(t_i)$$

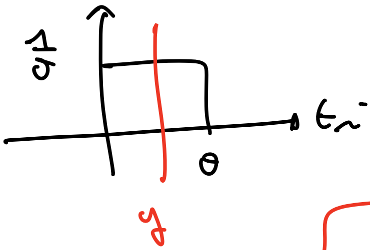


$$\boxed{\hat{\theta}_{MV} = \max_{i \in \{1, \dots, n\}} T_i}$$

$E(\hat{\theta}_{MV})$?

$$P[\sup T_i < y] = P(T_1 < y, \dots, T_n < y]$$
$$\underset{T_1, \dots, T_n \text{ ind}}{=} \prod_{i=1}^{n} P(T_i < y]$$

$T_i \sim U(]0,\theta[)$ donc $P(T_i < y) = \begin{cases} 0 & y \leq 0 \\ 1 & y \geq \theta \\ \underbrace{\int_0^y \frac{1}{\theta} dt_i}_{\frac{y}{\theta}} & y \in]0,\theta[\end{cases}$



$$\boxed{P(\underbrace{\sup T_i}_{Y_n} < y) = \begin{cases} 0 & y \leq 0 \\ 1 & y \geq \theta \\ \left(\frac{y}{\theta}\right)^n & y \in]0,\theta[\end{cases}}$$

La densité de $\sup T_i$ est donc

$$p(y_n) = \begin{cases} 0 & y \notin]0,\theta[\\ \frac{n y^{n-1}}{\theta^n} & y \in]0,\theta[\end{cases}$$

$$P[X < x] = \int_{-\infty}^{x} p(u)\,du$$

<u>Moyenne</u> $E(Y_n) = \int_0^\theta y \frac{n y^{n-1}}{\theta^n} dy$

$$= \frac{n}{\theta^n} \left[\frac{y^{n+1}}{n+1}\right]_0^\theta$$

$$= \theta \frac{n}{n+1} \neq \theta$$

donc Y_n est biaisé

Mais $\boxed{\frac{n+1}{n} Y_n = \hat{\theta}_2 \text{ est bien un estimateur non biaisé de } \theta}$

Pour comparer $\hat{\theta}_1$ et $\hat{\theta}_2$, il faut calculer $\mathrm{Var}(\hat{\theta}_2)$

$$\mathrm{Var}[\hat{\theta}_2] = \mathrm{Var}\left[\frac{n+1}{n} Y_n\right] = \left(\frac{n+1}{n}\right)^2 \mathrm{Var}\, Y_n$$

$$\mathrm{Var}[Y_n] = E[Y_n^2] - \underbrace{E^2[Y_n]}_{\left(\theta \frac{n}{n+1}\right)^2}$$

$$E[Y_n^2] = \int_0^\theta y^2 \frac{n y^{n-1}}{\theta^n} dy = \frac{n}{\theta^n} \frac{\theta^{n+2}}{n+2}$$

$$= \theta^2 \frac{n}{n+2}$$

$$\mathrm{Var}(Y_n) = \left[\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right] \theta^2$$

$$= n \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)^2} \theta^2$$

$$= \boxed{\frac{n\,\theta^2}{(n+2)(n+1)^2}}$$

donc $\boxed{\mathrm{Var}\,\hat{\theta}_2 = \frac{\theta^2}{n(n+2)}}$

à comparer avec $\boxed{\text{Var } \hat{\theta}_1 = \frac{\theta^2}{3n}}$

On préfère l'estimateur $\hat{\theta}_2$ qui a une variance plus faible pour n grand !!

Cours du 5/02/2024

$X_1, \dots, X_n$ échantillon de loi $\Gamma(v, \theta)$
Estimateur des moments de $\binom{v}{\theta}$?

TABLES

$$\begin{cases} E[X_i] = \dfrac{v}{\theta} \\ E[X_i^2] = \operatorname{Var} X_i + E^2[X_i] = \dfrac{v}{\theta^2} + \dfrac{v^2}{\theta^2} \end{cases}$$

Inversion de ce système

$$\begin{cases} v = \theta\, E[X_i] \\ E[X_i^2] = \dfrac{E[X_i]}{\theta} + E^2[X_i] \end{cases} \iff \begin{cases} \left(E[X_i^2] - E^2[X_i]\right)\theta = E[X_i] \\ \text{donc } \boxed{\theta = \dfrac{E[X_i]}{E[X_i^2] - E^2[X_i]}} \\ \boxed{v = \dfrac{E^2[X_i]}{E[X_i^2] - E^2[X_i]}} \end{cases}$$

Estimateurs des moments

$$\hat{\theta}_{Mo} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum x_i\right)^2}$$

$$\hat{v}_{Mo} = \frac{\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2}$$

BAYES

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

$$\underbrace{p(\theta \mid x_1, \dots, x_n)}_{\boxed{\text{Loi a posteriori de } \theta}} = \frac{\overbrace{p(x_1, \dots, x_n \mid \theta)}^{\text{vraisemblance}}\ \overbrace{p(\theta)}^{\text{loi a priori}}}{\underbrace{p(x_1, \dots, x_n)}_{\text{"constante" de normalisation}}}$$

$$\propto p(x_1, \dots, x_n \mid \theta)\; p(\theta)$$

proportionnel à

MMSE = minimum mean square error.

$\hat{\theta}_{MMSE} =$ moyenne de la loi a posteriori qui minimise $E[(\hat{\theta}-\theta)^2]$

$= E[\theta \mid x_1, \dots, x_n]$

Exemple

$x_i \sim N(\theta, \sigma^2) \Rightarrow$ vraisemblance

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i-\theta)^2}{2\sigma^2}\right] \quad (*)$$

Loi apriori $\qquad \theta \sim N(\mu, v^2) \qquad \mu$ et v connus

$$p(\theta) = \frac{1}{\sqrt{2\pi v^2}} \exp\left[\frac{(\theta-\mu)^2}{2v^2}\right] \quad (*)$$

Estimateurs MAP et MMSE de θ ?

Loi aposteriori

$$p(\theta / x_1, \dots x_n) \propto \boxed{\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2\right] \times \exp\left[-\frac{(\theta-\mu)^2}{2v^2}\right]}$$

$$\ln p(\theta / x_1, \dots, x_n) = K - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2 - \frac{1}{2v^2}(\theta-\mu)^2$$

$$\frac{\partial \ln p}{\partial \theta} = 0 \iff -\frac{1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}\cancel{2}(x_i-\theta)(-1) - \frac{1}{\cancel{2}v^2}\cancel{2}(\theta-\mu) = 0$$

$$\iff \underbrace{\frac{\theta}{v^2} + \frac{n\theta}{\sigma^2}}_{} = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i + \frac{\mu}{v^2}$$

$$\theta\left(\frac{1}{v^2} + \frac{n}{\sigma^2}\right)$$

$$\theta\left(\frac{\sigma^2 + nv^2}{v^2\sigma^2}\right)$$

d'où

$$\boxed{\theta = \frac{v^2}{\sigma^2 + nv^2}\sum_{i=1}^{n} x_i + \frac{\sigma^2}{\sigma^2+nv^2}\,\mu}$$

$1 - \alpha_n$

donc

$$\boxed{\hat{\theta}_{MAP} = \frac{nv^2}{nv^2+\sigma^2}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)}_{} + \frac{\sigma^2}{\sigma^2+nv^2}\,\mu}$$

α_n $\underbrace{\hat\theta_{MV}}_{\text{vient des données}}$ vient de la loi a priori

n "grand" $n \to +\infty$ $\alpha_n \to 1$ $1-\alpha_n \to 0$ $\hat\theta_{MAP} \approx \hat\theta_{MV}$

On fait confiance aux données !

n "petit" $n \to 0$ $\alpha_n \to 0$ $1-\alpha_n \to 1$ $\hat\theta_{MAP} \approx \mu$

on fait confiance à l'information a priori !

$\sigma^2 \to 0$ $\alpha_n \approx 1$ $1-\alpha_n \approx 0$ on fait confiance aux données !

$\sigma^2 \to +\infty$ $\alpha_n \approx 0$ $1-\alpha_n \approx 1$ on fait confiance à l'info a priori !

Estimateur MMSE

$$\hat\theta_{MMSE} = E[\theta \mid x_1,\dots,x_n]$$

TD du 5/02

$$f(x) = \begin{cases} \frac{\alpha}{\theta} x^{\alpha-1} \exp(-\frac{x^\alpha}{\theta}) & x>0 \\ 0 & x \leq 0 \end{cases}$$

1) Loi de $U = X^\alpha$ $\iff$ $X = U^{1/\alpha}$

$$g(u) = \frac{\alpha}{\theta} u^{\frac{\alpha-1}{\alpha}} \exp\left(-\frac{u}{\theta}\right) \underbrace{\left|\frac{dx}{du}\right|}_{\frac{1}{\alpha} u^{1/\alpha - 1}}$$

$$\boxed{g(u) = \frac{1}{\theta} \exp\left(-\frac{u}{\theta}\right)}$$

Domaine de U $x>0 \iff u^{1/\alpha} > 0 \iff u>0$

donc $\boxed{g(u) = \frac{1}{\theta} \exp\left(-\frac{u}{\theta}\right) \mathbb{1}_{\mathbb{R}^+}(u)}$

$$U \sim \Gamma(\underbrace{1}_{a}, \underbrace{\tfrac{1}{\theta}}_{b})$$

$$\boxed{\begin{aligned} E[U] &= \frac{a}{b} = \theta \\ Var[U] &= \frac{a}{b^2} = \theta^2 \end{aligned}}$$

2) Estimateur du max de vraisemblance de θ

vraisemblance $\quad L(x_1,\dots,x_n;\theta) = \prod_{i=1}^{n} \left[\frac{\lambda}{\theta} x_i^{\lambda-1} \exp\left(-\frac{x_i^\lambda}{\theta}\right)\right]$

Log vraisemblance $\quad \ln L = \sum_{i=1}^{n} \left[\ln\lambda - \ln\theta + \ln\left(x_i^{\lambda-1}\right) - \frac{x_i^\lambda}{\theta} \right]$

max de vraisemblance $\quad \frac{\partial \ln L}{\partial\theta} \overset{\geq 0}{=0} \iff -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i^\lambda \overset{\geq 0}{=0}$

$\iff \boxed{\theta \overset{\leq}{=} \frac{1}{n}\sum_{i=1}^{n} x_i^\lambda}$

donc $\quad \boxed{\hat{\theta}_{MV} = \frac{1}{n}\sum_{i=1}^{n} x_i^\lambda}$

θ	0		$\frac{1}{n}\sum_{i=1}^{n} x_i^\lambda$		$+\infty$
$\frac{\partial \ln L}{\partial\theta}$		$+$	0	$-$	
$\ln L$		↗		↘	

Estimateur sans biais? convergent? efficace?

Biais $\quad b_n(\theta) = E[\hat{\theta}] - \theta$

$E(\hat{\theta}) = E\left[\frac{1}{n}\sum x_i^\lambda\right] = \frac{1}{n}\sum_{i=1}^{n} E[\underbrace{x_i^\lambda}_{U_i}]$

$= \frac{1}{n}\sum_{i=1}^{n} \theta = \boxed{\theta}$

donc $\hat{\theta}$ non biaisé

Variance $\quad Var[\hat{\theta}] = Var\left[\frac{1}{n}\sum_{i=1}^{n} x_i^{d}\right]$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(\underbrace{x_i^{d}}_{U_i}) = \frac{1}{n^2}\sum_{i=1}^{n} \theta^2$$

$$= \boxed{\frac{\theta^2}{n}}$$

$x_1, \dots, x_n$ ind

$\Rightarrow x_1^{d}, \dots, x_n^{d}$ ind

$$\boxed{\left.\begin{array}{l} Biais(\hat{\theta}) = 0 \\ Var\,\hat{\theta} \underset{n\to+\infty}{\longrightarrow} 0 \end{array}\right\} \Rightarrow \hat{\theta} \text{ convergent}}$$

Efficacité $\quad BCR = \dfrac{1}{E\left[-\dfrac{\partial^2 \ln L}{\partial \theta^2}\right]}$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i^{d}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i^{d}$$

$$E[\ ] = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} \underbrace{E(x_i^{d})}_{\theta} = \frac{n}{\theta^2} - \frac{2n}{\theta^2} = -\frac{n}{\theta^2}$$

donc $\boxed{BCR = \frac{\theta^2}{n}}$

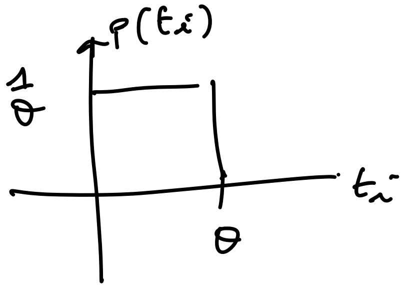
$$\left\{\begin{array}{l} \hat{\theta} \text{ non biaisé} \\ Var\,\hat{\theta} = BCR \end{array}\right. \Rightarrow \boxed{\hat{\theta} \text{ est l'estimateur efficace de } \theta}$$

Erreur quadratique moyenne

$$E[(\hat{\theta}-\theta)^2] = \text{Biais}^2 + \text{Variance} = \boxed{\frac{\theta^2}{n}}$$

Exo 1

1)



$E(T_i) = \dfrac{\theta}{2}$

$\underbrace{Var(T_i) = \dfrac{\theta^2}{12}}_{\text{Tables}}$

2) $\bar{T} = \dfrac{1}{n}\sum_{i=1}^{n} T_i \quad (= \hat{\theta})$

$$E(\bar{T}) = \frac{1}{n}\sum_{i=1}^{n} \underbrace{E(T_i)}_{\frac{\theta}{2}} = \frac{\theta}{2}$$

donc $E(\bar{T}) - \theta = \dfrac{\theta}{2} - \theta = -\dfrac{\theta}{2} \neq 0$

donc $\bar{T}$ estimateur <u>biaisé</u> de θ

$\hat{\theta} = 2\bar{T} \qquad E(\hat{\theta}) = 2 \times \dfrac{\theta}{2} = \theta$

<u>$\hat{\theta}$ non biaisé</u>

$$Var\,\hat{\theta} = 4\,Var(\bar{T}) = 4\,Var\left(\frac{1}{n}\sum T_i\right)$$

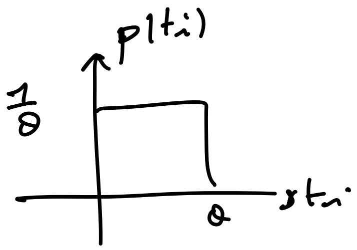
$$= \frac{4}{n^2}\sum_{i=1}^{n} \underbrace{Var\,T_i}_{\frac{\theta^2}{12}}$$

$$= \boxed{\frac{\theta^2}{3n}}$$

$$\begin{cases} \hat{\theta} \text{ non biaisé} \\ Var\,\hat{\theta} \xrightarrow[n\to+\infty]{} 0 \end{cases} \quad \text{donc} \quad \boxed{\hat{\theta} \text{ estimateur convergent}}$$

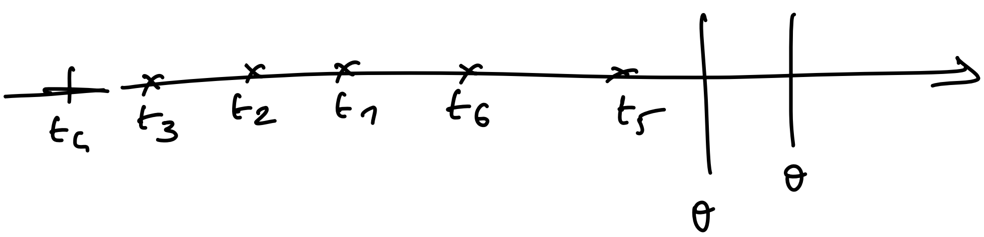
3) <u>Estimateur du max de vraisemblance</u>

$$L(t_1, \dots, t_n ; \theta) = \prod_{i=1}^{n} \left[\frac{1}{\theta} \mathbb{1}_{]0,\theta[}(t_i) \right]$$



$$= \begin{cases} \frac{1}{\theta^n} & \text{si tous les } t_i \in]0,\theta[\\ 0 & \text{sinon} \end{cases}$$

$$p(t_i) = \frac{1}{\theta} \mathbb{1}_{]0,\theta[}(t_i)$$



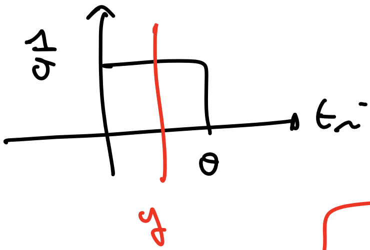
$$\boxed{\hat{\theta}_{MV} = \max_{i\in\{1,\dots,n\}} T_i}$$

$E(\hat{\theta}_{MV})$?

$$P[\sup T_i < y] = P(T_1 < y, \dots, T_n < y]$$

$$\underset{T_1,\dots,T_n \text{ iid}}{=} \prod_{i=1}^{n} P(T_i < y]$$

$T_i \sim U(]0,\theta[)$ donc $P(T_i<y) = \begin{cases} 0 & y \le 0 \\ 1 & y \ge \theta \\ \underbrace{\int_0^y \frac{1}{\theta}\,dt_i}_{\frac{y}{\theta}} & y \in]0,\theta[\end{cases}$



$$P(\underbrace{\sup T_i}_{Y_n} < y) = \begin{cases} 0 & y \le 0 \\ 1 & y \ge \theta \\ \left(\frac{y}{\theta}\right)^n & y \in]0,\theta[\end{cases}$$

La densité de Sup T_i est donc

$$p(y_n) = \begin{cases} 0 & y \notin]0,\theta[\\ \frac{n y^{n-1}}{\theta^n} & y \in]0,\theta[\end{cases}$$

$$P[X<x] = \int_{-\infty}^{x} p(u)\,du$$

Moyenne $E(Y_n) = \int_0^\theta y \frac{n y^{n-1}}{\theta^n}\,dy$

$$= \frac{n}{\theta^n}\left[\frac{y^{n+1}}{n+1}\right]_0^\theta$$

$$= \theta \frac{n}{n+1} \neq \theta$$

donc Y_n est biaisé

Mais $\frac{n+1}{n} Y_n = \hat{\theta}_2$ est bien un estimateur non biaisé de θ

Pour comparer $\hat{\theta}_1$ et $\hat{\theta}_2$, il faut calculer $Var(\hat{\theta}_2)$

$$\mathrm{Var}[\hat{\theta}_2] = \mathrm{Var}\left[\frac{n+1}{n} Y_n\right] = \left(\frac{n+1}{n}\right)^2 \mathrm{Var}\, Y_n$$

$$\mathrm{Var}(Y_n) = E(Y_n^2) - \underbrace{E^2(Y_n)}_{\left(\theta \frac{n}{n+1}\right)^2}$$

$$\int_0^\theta y^2 \frac{n y^{n-1}}{\theta^n}\, dy = \frac{n}{\theta^n} \frac{\theta^{n+2}}{n+2}$$

$$= \theta^2 \frac{n}{n+2}$$

$$\mathrm{Var}(Y_n) = \left[\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right] \theta^2$$

$$= n \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)^2} \theta^2$$

$$= \boxed{\frac{n\,\theta^2}{(n+2)(n+1)^2}}$$

donc $\boxed{\mathrm{Var}\, \hat{\theta}_2 = \dfrac{\theta^2}{n(n+2)}}$

à comparer avec $\boxed{Var\ \hat{\theta}_1 = \frac{\theta^2}{3n}}$

On préfère l'estimateur $\hat{\theta}_2$ qui a une variance plus faible pour n grand !!

COURS du 12/02/2024

H_0 pas d'anomalie

H_1 anomalie

$\frac{1}{n}\sum x_i = T > S_\alpha$

$\alpha = P[\text{Rejeter } H_0 \mid H_0 \text{ vraie}] = P[\underbrace{\text{décider qu'il y a une anomalie}} \mid \text{il n'y a pas d'anomalie}]$

$= PFA$ probabilité de fausse alarme

$\beta = P[\text{Rejeter } H_1 \mid H_1 \text{ vraie}] = P[\underbrace{\text{décider qu'il n'y a pas d'anomalie}}_{T < S_\alpha} \mid \text{il y a une anomalie}]$

$= PND$ probabilité de non détection

Exercice $X_i \sim N(m, \sigma^2)$ σ^2 connue

H_0 $m = m_0$

H_1 $m = m_1 > m_0$

Rejet de H_0 si $\frac{1}{n}\sum_{i=1}^n x_i > S_\alpha$

Comment déterminer S_α ?

On se fixe α (= 0.01 ou 0.05) $= P[T = \frac{1}{n}\sum_{i=1}^n x_i > S_\alpha \mid m = m_0]$

$= 1 - P[\frac{1}{n}\sum_{i=1}^n x_i \leq S_\alpha \mid m = m_0]$ //

Rq $\frac{1}{n}\sum_{i=1}^n x_i = (\frac{1}{n}, \dots, \frac{1}{n}) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$X_1 \dots X_n$ ind
$X_i \sim \mathcal{N}(m, \sigma^2)$ $\Rightarrow$ $\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ vecteur gaussien

$\Rightarrow \underbrace{A}_{T} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ de loi gaussienne

$rg(A) = 1$ max

$$\boxed{\frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}\Big(\underbrace{m}_{\frac{1}{n}\sum \underbrace{E[X_i]}_{m}}, \underbrace{\frac{\sigma^2}{n}}_{\frac{1}{n^2}\sum_{i=1}^{n} \underbrace{var(X_i)}_{\sigma^2}}\Big)}$$

$$U = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - m}{\sqrt{\sigma^2/n}} \sim \mathcal{N}(0,1)$$

$$\alpha = 1 - P\Big[\frac{1}{n}\sum_{i=1}^{n} X_i \leq S_\alpha \,\Big|\, \underbrace{H_0 \text{ vraie}}_{m = m_0}\Big] \quad (*)$$

$$= 1 - P\Big[\underbrace{\frac{\frac{1}{n}\sum X_i - m}{\sqrt{\sigma^2/n}}}_{U \sim \mathcal{N}(0,1)} \leq \frac{S_\alpha - m}{\sqrt{\sigma^2/n}} \,\Big|\, m = m_0\Big]$$

$$\boxed{\alpha = 1 - F_{\mathcal{N}(0,1)}\left[\frac{S_\alpha - m_0}{\sqrt{\sigma^2/n}}\right]} \quad (*)$$

$F_{\mathcal{N}(0,1)}[x] = 1 - \alpha$ donc $\dfrac{S_\alpha - m_0}{\sqrt{\sigma^2/n}} = F^{-1}_{\mathcal{N}(0,1)}(1-\alpha)$

donc $\boxed{S_\alpha = m_0 + \sqrt{\frac{\sigma^2}{n}}\, F^{-1}_{\mathcal{N}(0,1)}(1-\alpha)}$

<u>1re étape</u> : seuil S_α

<u>Seconde étape</u> : $PND = \beta$?

$$PND = P\Big[\underbrace{\text{Rejeter } H_1}_{\frac{1}{n}\sum_{i=1}^{n} X_i \leq S_\alpha} \,\Big|\, \underbrace{H_1 \text{ vraie}}_{m = m_1}\Big] \quad (> m_0)$$

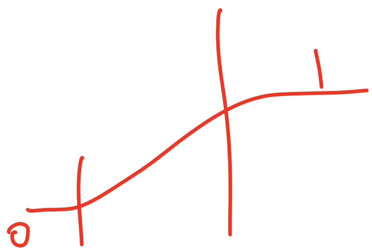
Rq $PD = \pi = 1 - PND = P\Big[\frac{1}{n}\sum X_i > S_\alpha \,\Big|\, m = m_1\Big]$

On en déduit, en changeant m_0 en m_1 dans (*)

$$\boxed{\Pi = 1 - F_{N(0,1)}\left[\frac{S_\alpha - m_1}{\sqrt{\frac{\sigma^2}{n}}}\right]}$$

ou

$$\boxed{\beta = F_{N(0,1)}\left(\frac{S_\alpha - m_1}{\sqrt{\frac{\sigma^2}{n}}}\right)}$$



<u>COURBES COR</u> Π en fonction de α

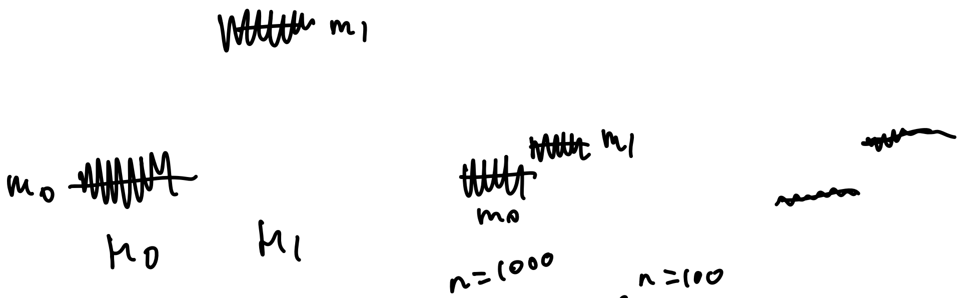
$$\left|\begin{array}{l} \Pi = 1 - F\left[\dfrac{S_\alpha - m_1}{\sigma/\sqrt{n}}\right] \\ S_\alpha = m_0 + \sqrt{\dfrac{\sigma^2}{n}}\, F^{-1}(1-\alpha) \end{array}\right.$$

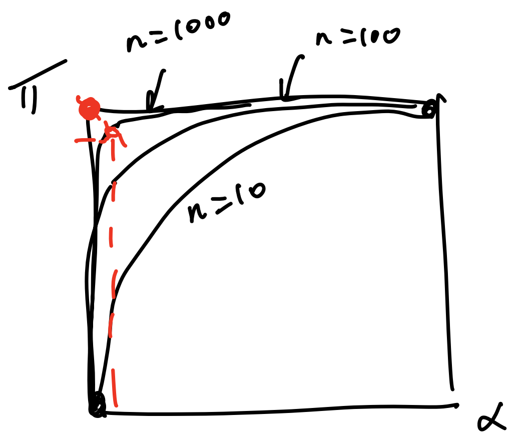
$$\boxed{\Pi = 1 - F\left[\underbrace{\frac{m_0 - m_1}{\sigma/\sqrt{n}}}_{-\frac{(m_1 - m_0)}{\sigma/\sqrt{n}}} + F^{-1}(1-\alpha)\right]}$$

Ce qui est important pour le test est la valeur de

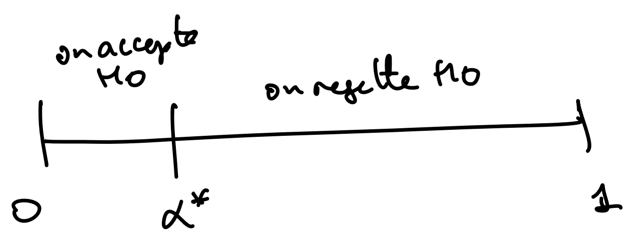
$$\sqrt{n}\,\frac{(m_1 - m_0)}{\sigma}$$

Π est d'autant plus grand que $\dfrac{\sqrt{n}\,(m_1 - m_0)}{\sigma}$ est grand





p-valeur = plus petite valeur de α pour laquelle on rejette H_0 $= \alpha^*$



NEYMAN PEARSON

Rejet de H_0 si $\dfrac{L(x_1,\dots,x_n | H_1)}{L(x_1,\dots,x_n | H_0)} > S_\alpha$

$\underline{\text{Ex}}$

$x_i \sim N(m, \sigma^2)$

$H_0 \quad m = m_0$

$H_1 \quad m = m_1 > m_0$

Neyman Pearson?

On rejette H_0 si

$$\frac{\prod_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left[-\frac{(x_i - m_1)^2}{2\sigma^2}\right]}{\prod_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left[-\frac{(x_i - m_0)^2}{2\sigma^2}\right]} > S_\alpha$$

Un test équivalent est

Rejet de H_0 si

(on prend le Ln)

$$\sum_{i=1}^{n} \left[-\frac{(x_i - m_1)^2}{2\sigma^2} + \frac{(x_i - m_0)^2}{2\sigma^2} \right] > \underbrace{\ln S_\alpha}_{S'_\alpha}$$

$$\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left[-(\cancel{x_i^2} - 2m_1 x_i + m_1^2) + \cancel{x_i^2} - 2m_0 x_i + m_0^2 \right] > S'_\alpha$$

$$2(m_1 - m_0) \sum_{i=1}^{n} x_i + n(m_0^2 - m_1^2) > 2\sigma^2 S'_\alpha$$

$m_1 > m_0 \Rightarrow m_1 - m_0 > 0$

$$\boxed{\text{Rejet de } H_0 \text{ si } \sum_{i=1}^{n} x_i > A_\alpha}$$

stat de test $\quad T = \sum_{i=1}^{n} X_i$

zone critique $\quad \left\{ (x_1, \ldots, x_n) / \sum_{i=1}^{n} x_i > A_\alpha \right\}$

$\boxed{\text{TD du } 12/02/2024}$

1) vraisemblance $\prod_{i=1}^{n} \theta e^{-\theta t_i} = \boxed{\theta^n \exp\left(-\theta \sum_{i=1}^{n} t_i\right)}$

Log vraisemblance $\ln L = n \ln \theta - \theta \sum_{i=1}^{n} t_i$

$$\frac{\partial \ln L}{\partial \theta} = 0 \iff \frac{n}{\theta} - \sum_{i=1}^{n} t_i = 0$$

$$\theta = \frac{n}{\sum_{i=1}^{n} t_i}$$

donc $\boxed{\hat{\theta}_{MV} = \frac{n}{\sum_{i=1}^{n} T_i}}$

Rq $\hat{\theta}_{MV} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} T_i}$

T_i est de densité $\theta e^{-\theta t_i}$ $t_i \geq 0$

$$E(T_i) = \frac{1}{\theta}$$

donc $\theta = \frac{1}{E(T_i)} \approx \frac{1}{\frac{1}{n}\sum T_i}$

c'est logique !!

Est-ce que c'est bien un maximum ?

$$\frac{\partial \ln L}{\partial \theta} \geq 0 \iff \frac{n}{\theta} - \sum_{i=1}^{n} t_i \geq 0$$

$\Leftrightarrow \quad \theta \leq \dfrac{n}{\sum t_i}$

θ	0		$\dfrac{n}{\sum t_i}$		$+\infty$
$\dfrac{\partial \ln L}{\partial \theta}$		$+$	0	$-$	
L		$\nearrow$		$\searrow$	

donc $\theta = \dfrac{n}{\sum t_i}$ est le maximum global unique de la vraisemblance !

② $\boxed{\hat{\theta}_{MAP} \text{ et } \hat{\theta}_{MMSE} \text{ ?}}$

Loi a posteriori de l'échantillon $(t_1, \dots, t_n)$

$$p(\theta \mid t_1, \dots, t_n) \propto \underbrace{p(t_1, \dots, t_n \mid \theta)}_{\text{vraisemblance}} \times \underbrace{p(\theta)}_{\text{loi a priori}}$$

$$\propto \theta^n \exp\left(-\theta \sum_{i=1}^{n} t_i\right) \lambda e^{-\lambda\theta}$$

proportionnel.

$$\propto \boxed{\theta^n \exp\left[-\theta\left(1 + \sum_{i=1}^{n} t_i\right)\right]}$$

donc $\boxed{\hat{\theta}_{MAP} = \dfrac{n}{\sum_{i=1}^{n} T_i + 1}}$

Estimateur MMSE

$$\hat{\theta}_{MMSE} = E[\theta \mid T_1, \dots, T_n]$$

$$\boxed{\theta \mid t_1, \dots, t_n \sim \Gamma\Big(\underbrace{n+1}_{\nu},\ \underbrace{\sum_{i=1}^{n} t_i + \lambda}_{\text{"}\theta\text{"}}\Big)}$$

$$E[X] = \frac{\nu}{\theta}$$

donc
$$\boxed{E[\theta \mid T_1, \dots, T_n] = \frac{n+1}{\sum_{i=1}^{n} T_i + \lambda}}$$

Résumé

$$\hat{\theta}_{MV} = \frac{n}{\sum_{i=1}^{n} T_i} = \frac{1}{\overline{T}}$$

$$\hat{\theta}_{MAP} = \frac{n}{\sum_{i=1}^{n} T_i + \lambda} = \frac{1}{\overline{T} + \frac{\lambda}{n}}$$

$$\hat{\theta}_{MMSE} = \frac{n+1}{\sum_{i=1}^{n} T_i + \lambda}$$

$$= \frac{1 + \frac{1}{n}}{\overline{T} + \frac{\lambda}{n}}$$

Notation

$$\overline{T} = \frac{1}{n} \sum_{i=1}^{n} T_i$$

Commentaire : $\hat{\theta}_{MAP} \underset{n \to \infty}{\sim} \hat{\theta}_{MV}$

quand le nombre de données augmente, on fait confiance aux données

Exo 2

$$z_1 \sim N(\theta, \sigma^2)$$
$$z_2 \sim N(\theta, \sigma^2)$$

$$L(z_1, z_2 ; \theta) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(z_1-\theta)^2}{2\sigma^2} - \frac{(z_2-\theta)^2}{2\sigma^2}\right]$$

z_1 et z_2 ind

$$\hat{\theta}_{MV} = 0 \iff L(z_1, z_2 ; \theta = 0) > L(z_1, z_2 ; \theta = 1)$$

$$\iff \cancel{\frac{1}{2\pi\sigma^2}} e^{-\frac{z_1^2+z_2^2}{2\sigma^2}} > \cancel{\frac{1}{2\pi\sigma^2}} e^{-\frac{(z_1-1)^2+(z_2-1)^2}{2\sigma^2}}$$

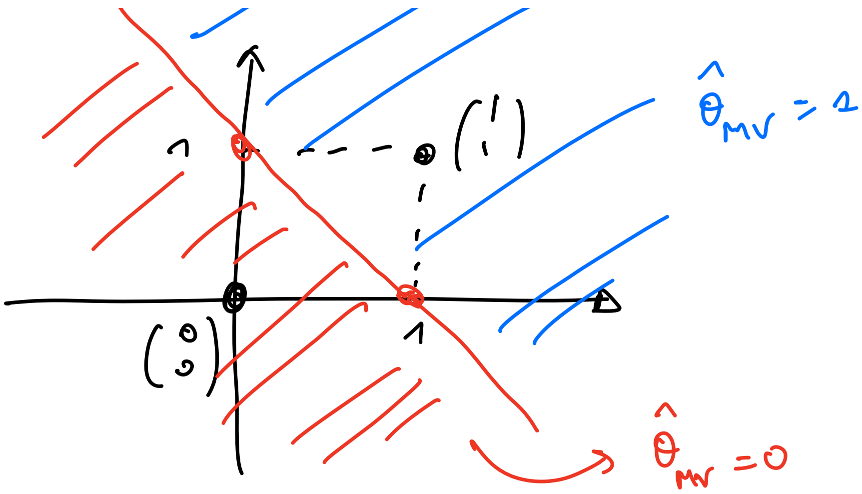
$\iff$ (on prend le Log)

$$-\frac{1}{\cancel{2\sigma^2}}(z_1^2+z_2^2) > -\frac{1}{\cancel{2\sigma^2}}\left((z_1-1)^2+(z_2-1)^2\right)$$

$$\iff -\cancel{z_1^2} - \cancel{z_2^2} \geqslant -(\cancel{z_1^2} - 2z_1 + 1) - (\cancel{z_2^2} - 2z_2 + 1)$$

$$\iff 2(z_1 + z_2 - 1) \leq 0$$

$$\iff \boxed{z_1 + z_2 \leq 1}$$



3) $p(\theta) = p^{1-\theta}(1-p)^{\theta}$ $\begin{cases} p & \text{si } \theta = 0 \\ 1-p & \text{si } \theta = 1 \end{cases}$

Loi a priori

<u>Loi a posteriori</u>

$$L(z_1, z_2 \mid \theta) \times p(\theta)$$

$$\parallel$$

$$\frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(z_1-\theta)^2 + (z_2-\theta)^2\right]\right\} \times p^{1-\theta}(1-p)^{\theta}$$

$$\hat{\theta}_{MAP} = 0 \iff \exp\left[-\frac{1}{2\sigma^2}(z_1^2+z_2^2)\right] p \geq \exp\left[-\frac{1}{2\sigma^2}\left((z_1-1)^2+(z_2-1)^2\right)\right] \times (1-p)$$

$$\iff \cancel{-\frac{1}{2\sigma^2}(z_1^2+z_2^2)} + \ln p \geq -\frac{1}{2\sigma^2}\Big[\underbrace{(z_1-1)^2+(z_2-1)^2}_{z_1^2+z_2^2-2z_1-2z_2+2}\Big] + \ln(1-p)$$

$$\hat{\theta}_{MAP} = 0 \iff \ln\left(\frac{p}{1-p}\right) \geq \frac{z_1 + z_2 - 1}{\sigma^2}$$

$$\iff \boxed{z_1 + z_2 \leq 1 + \sigma^2 \ln\left(\frac{p}{1-p}\right)}$$