Cours #1 Stat 1SN – 16/10/2023

Exemple 1

$X_i \sim \mathcal{N}(m, \sigma^2)$ $\quad \theta = (m)$

σ^2 connue

$$\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\tilde{\theta}_n = \frac{2}{n(n+1)}\sum_{i=1}^{n} i X_i$$

Lequel de ces deux estimateurs préférez vous ?

Biais.

$\theta = m$

$$E[\hat{\theta}_n] = \frac{1}{n}\sum_{i=1}^{n} \underbrace{E(X_i)}_{m} = m$$

donc $\hat{\theta}_n$ estimateur sans biais de m $\quad (E(\hat{\theta}_n) - m = 0)$

$$E[\tilde{\theta}_n] = \frac{2}{n(n+1)}\sum_{i=1}^{n} \underbrace{E[iX_i]}_{im}$$

$$= \frac{2m}{n(n+1)}\underbrace{\sum_{i=1}^{n} i}_{\frac{n(n+1)}{2}} = m$$

$E[ax+b] = aE[X]+b$

$\tilde{\theta}_n$ est aussi un estimateur non biaisé de m

Variance

$Var[ax+b] = a^2 Var(X)$

si X et Y sont ind

alors $Var(X+Y) = Var X + Var Y$

$$Var(\hat{\theta}_n) = \frac{1}{n^2}\sum_{i=1}^{n} \underbrace{Var(X_i)}_{\sigma^2} = \frac{\sigma^2}{n}$$

$X_1, \dots, X_n$ ind

Rq : $\hat{\theta}_n$ sans biais et $var(\hat{\theta}_n) \xrightarrow[n\to+\infty]{} 0$

donc $\hat{\theta}_n$ est convergent

$$Var(\tilde{\theta}_n) = Var\left[\frac{2}{n(n+1)} \sum_{i=1}^{n} i x_i\right]$$

$$\boxed{\sum_{i=1}^{n} i^2 = 1^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$= \frac{4}{n^2(n+1)^2} \sum_{i=1}^{n} \underbrace{Var(i x_i)}_{i^2 Var x_i = i^2 \sigma^2}$$

$$= \frac{4\sigma^2}{n^{\cancel{2}}(n+1)^{\cancel{2}}} \frac{\cancel{n}\cancel{(n+1)}(2n+1)}{6}$$

$$= \boxed{\frac{2}{3}\sigma^2 \frac{2n+1}{n(n+1)}} \underset{n \to +\infty}{\sim} \frac{4\sigma^2}{3n} > \frac{\sigma^2}{n}$$

$Var\, \hat{\theta}_n < Var\, \tilde{\theta}_n$

$Biais(\hat{\theta}_n) = Biais(\tilde{\theta}_n) = 0$ donc $\boxed{\text{on préfèrera } \hat{\theta}_n \text{ à } \tilde{\theta}_n}$

$Biais\, \hat{\theta}_1 = 0.2$ $Biais(\hat{\theta}_2) = 0{,}02$

$Var\, \hat{\theta}_1 = 1$ $Var(\hat{\theta}_2) = 2$

erreur quadratique moyenne

$e_1(\theta) = (0.2)^2 + 1$ $e_2(\theta) = (0{,}02)^2 + 2$

$= 1.04$

ici $e_1(\theta) < e_2(\theta)$ donc on préfère $\hat{\theta}_1$ à $\hat{\theta}_2$

Preuve de $e_n(\theta) = \sigma_n(\theta) + b_n^2(\theta)$

$$e_n(\theta) = E\left[(\hat{\theta}_n - \theta)^2\right] = E\left[\left(\hat{\theta}_n - E[\hat{\theta}_n] + E[\hat{\theta}_n] - \theta\right)^2\right]$$

$$= \underbrace{E\left[\left(\hat{\theta}_n - E[\hat{\theta}_n]\right)^2\right]}_{\sigma_n(\theta)} + \underbrace{E\Big[\underbrace{\left(E[\hat{\theta}_n] - \theta\right)^2}_{b_n^2(\theta)}\Big]}_{E[b_n^2(\theta)] = b_n^2(\theta)}$$

$$+2\,E\Big[\underbrace{\big(\hat\theta_n - E[\hat\theta_n]\big)}\,b_n(\theta)\Big]$$

$$2b_n(\theta)\Big(E[\hat\theta_n] - E[\underbrace{E[\hat\theta_n]}_{E(\hat\theta_n)}]\Big) = 0$$

$\boxed{CQFD}$

Exemple 2

$X_i \sim N(m, \sigma^2)$ $\quad \theta = \begin{pmatrix} m \\ \sigma^2 \end{pmatrix}$

m et σ^2 inconnus

$\hat m = \bar X_n = \frac{1}{n}\sum_{i=1}^n x_i$ (moyenne arithmétique)

$\hat\sigma^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar X_n)^2$

Biais de $\hat\sigma^2$? (on sait que $E[\hat m] - m = 0$)

$$E(\hat\sigma^2) = \frac{1}{n}\sum_{i=1}^n \underbrace{E\big[(X_i - m + m - \bar X_n)^2\big]}$$

$$E[(X_i - m)^2] \quad (1) \quad + E[(m - \bar X_n)^2] \quad (2)$$

$$-2E\big[(\bar X_n - m)(X_i - m)\big] \quad (3)$$

(1) $= \boxed{\sigma^2}$

(2) $= E[(\bar X_n - m)^2] = var(\bar X_n)$ $\qquad E\big[\big(\bar X_n - \underbrace{E[\bar X_n]}_{m}\big)^2\big]$

$= \boxed{\frac{\sigma^2}{n}}$ (calculé à l'exemple 1)

(3) $= -2E\Big[\Big(\frac{1}{n}\sum_{k=1}^n X_k - \underset{\uparrow}{m}\Big)(X_i - m)\Big]$

$\frac{1}{n}\sum_{k=1}^n m$

$$= -\frac{2}{n}E\Big[\sum_{k=1}^n (X_k - m)(X_i - m)\Big]$$

$$= -\frac{2}{n}\sum^n E[(X_k - m)(X_i - m)]$$

$n_{k=n}$

$\hookrightarrow \mathrm{cov}(X_k, X_i) \quad k \neq i$

$\hookrightarrow \mathrm{Var}\, X_i \quad k = i$

X_k et X_i iid $\Rightarrow \mathrm{cov}(X_k, X_i) = 0$

donc ③ $\boxed{-\frac{2\sigma^2}{n}}$

$$① + ② + ③ = \sigma^2 + \frac{\sigma^2}{n} - \frac{2\sigma^2}{n}$$

$$= \sigma^2 - \frac{\sigma^2}{n} = \left(\frac{n-1}{n}\right)\sigma^2$$

On en conclut

$$E(\hat{\sigma}^2) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{n-1}{n}\sigma^2\right) = \frac{n-1}{n}\sigma^2$$

donc l'estimateur $\hat{\sigma}^2$ est un estimateur biaisé de σ^2

$$E\Big[\frac{n}{n-1}\hat{\sigma}^2\Big] = \sigma^2$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$$

Un estimateur non biaisé de la variance est

$$\boxed{\tilde{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}$$

Exemple de calcul de borne de CRAMÉR-RAO

$X_i \sim N(m, \sigma^2)$ σ^2 connue

$\theta = m.$

Que vaut BCR ?

vraisemblance

$$p(x_1, \dots, x_n ; \theta) = \prod_{i=1}^{n} p(x_i ; \theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i - m)^2}{2\sigma^2}\right]$$

$\theta = m$

$$p(x_1, \dots, x_n ; \theta) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \theta)^2\right]$$

Log-vraisemblance

$$\operatorname{Ln} p(x_1, \dots, x_n ; \theta) = -\frac{n}{2} \operatorname{Ln}(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \theta)^2$$

Dérivés.

$$\frac{\partial \operatorname{Ln} p}{\partial \theta} = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} 2(x_i - \theta)(-1) = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \theta)$$

$$\frac{\partial^2 \ln p}{\partial \theta^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (-1) = -\frac{n}{\sigma^2}$$

Espérance

$$E\left[-\frac{n}{\sigma^2}\right] = -\frac{n}{\sigma^2}$$

Borne de Cramér_Rao pour un estimateur non biaisé de $\theta = m$

$$CRB = \frac{(1+0)^2}{-\left(-\frac{n}{\sigma^2}\right)} = \boxed{\frac{\sigma^2}{n}}$$

$\mathrm{var}\,\hat{\theta} \geq \frac{\sigma^2}{n}$ pour tout estimateur non biaisé de $\theta = m$

Le meilleur estimateur de la moyenne dans le cas du modèle $X_i \sim N(m, \sigma^2)$ est

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$p(x_1 \dots x_n ; \theta) = \prod_{i=1}^{n} p(x_i ; \theta)$$

$$\ln p = \sum_{i=1}^{n} \ln p(x_i ; \theta)$$

$$\frac{\partial^2 \ln p}{\partial \theta^2} = \sum_{i=1}^{n} \frac{\partial^2 \ln p(x_i ; \theta)}{\partial \theta^2}$$

$$E(\quad) = \sum_{i=1}^{n} E\left[\frac{\partial^2 \ln p(X_i ; \theta)}{\partial \theta^2}\right]$$

$$= n \, E\left[\frac{\partial^2 \ln p(X_1 ; \theta)}{\partial \theta^2}\right]$$

les X_i ont la même loi

Cours du 25/10/2023

$$A = Cov(\hat{\theta}) = \begin{pmatrix} Var\,\hat{\theta}_1 & & \\ & \ddots & \\ & & Var\,\hat{\theta}_p \end{pmatrix} \geqslant I^{-1} = B$$

cov( )

$$I = \left[- E\left(\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j} \right) \right]_{i,j}$$

$$x^T (A - B)\, x \geqslant 0 \quad \forall x \in \mathbb{R}^n$$

si $x = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ alors $(1\ 0 \cdots 0) \begin{pmatrix} Var\,\hat{\theta}_1 - b_{11} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \geqslant 0$

$$(1\ 0\ 0 \cdots 0) \begin{pmatrix} Var\,\hat{\theta}_1 - b_{11} \\ | \end{pmatrix} \geqslant 0$$

$$Var\,\hat{\theta}_1 - b_{11} \geqslant 0$$

$$\boxed{Var\,\hat{\theta}_1 \geqslant b_{11}}$$

Exemple $\quad x_i \sim N(m, \sigma^2) \quad \theta = \begin{pmatrix} m \\ \sigma^2 \end{pmatrix} \quad p = 2$

vraisemblance

$$L(x_1 \dots x_n ; \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i - m)^2}{2\sigma^2}\right]$$

$$\boxed{x_i \sim N(m, \sigma^2)} \Rightarrow \quad = \boxed{\frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - m)^2\right]}$$

Log vraisemblance

$$\ln L = -\frac{n}{2\pi} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - m)^2$$

Dérivés premiers

$$\frac{\partial \ln L}{\partial m} = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} 2(x_i - m)(-1) = \boxed{\frac{1}{\sigma^2}\left[\left(\sum_{i=1}^{n} x_i\right) - nm\right]}$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i - m)^2$$

Dérivés seconds

$$\frac{\partial^2 \ln L}{\partial m^2} = -\frac{n}{\sigma^2} \quad \Rightarrow E\left[-\frac{\partial^2 \ln L}{\partial m^2}\right] = \frac{n}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial(\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6} \sum_{i=1}^{n} (x_i - m)^2$$

$$\Rightarrow E[\ldots] = -\frac{n}{2\sigma^4} + \frac{1}{\sigma^6} \sum_{i=1}^{n} \underbrace{E[(x_i - m)^2]}_{\sigma^2} = -\frac{n}{2\sigma^4} + \frac{n}{\sigma^4} = \frac{n}{2\sigma^4}$$

$$\frac{\partial^2 \ln L}{\partial m \partial \sigma^2} = \frac{\partial^2 \ln L}{\partial \sigma^2 \partial m} = -\frac{1}{\sigma^4}\left[\left(\sum_{i=1}^{n} x_i\right) - nm\right]$$

$$I = \left(E\left[-\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j}\right]\right) = \begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ ?\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$$

$$E\left[-\frac{\partial^2 \ln L}{\partial m \partial \sigma^2}\right] = -\frac{1}{\sigma^4}\left[\sum_{i=1}^{n} \underbrace{E[x_i]}_{m} - nm\right] = 0$$

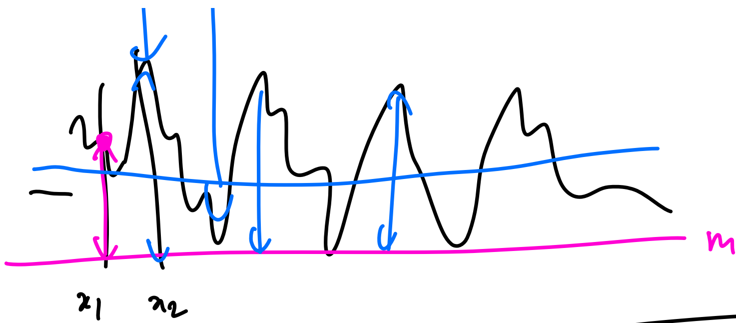
Inverse de I

$$B = I^{-1} = \begin{pmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{pmatrix}$$

CONCLUSION

$\cdot X_i \sim N(m, \sigma^2)$ $\Rightarrow$

$$\boxed{\begin{aligned} Var\,\hat{m} &\geq \frac{\sigma^2}{n} \\ Var\,\hat{\sigma}^2 &\geq \frac{2\sigma^4}{n} \end{aligned}}$$



$X_i \sim N(m, \sigma^2)$

$$L = k \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - m)^2\right]$$

Exemple 1

$X_i \sim P(\lambda)$ $\quad P[X_i = x_i] = \frac{\lambda^{x_i}}{x_i!} e^{-\lambda}$ $\quad x_i \in \mathbb{N}$

Quel est l'estimateur du max de vraisemblance de λ ?

vraisemblance

$$L(x_1, \dots, x_n; \lambda) = \prod_{i=1}^{n}\left[\frac{\lambda^{x_i}}{x_i!} e^{-\lambda}\right]$$

$$= \boxed{\frac{\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} e^{-n\lambda}}$$

Log vraisemblance

$$\ln L = \left(\sum_{i=1}^{n} x_i\right) \ln \lambda - \ln\left(\prod_{i=1}^{n} x_i!\right) - n\lambda$$

$$\frac{\partial \ln L}{\partial \lambda} \geqslant 0 \iff \left(\sum_{i=1}^{n} x_i\right)\frac{1}{\lambda} + 0 - n \geqslant 0$$

$$\iff \lambda \leq \frac{1}{n}\sum_{i=1}^{n} x_i$$

λ	0		$\frac{1}{n}\sum x_i$		$+\infty$
$\frac{\partial \ln L}{\partial \lambda}$		$+$	0	$-$	
$\ln L$	$-\infty$	$\nearrow$		$\searrow$	$-\infty$

$\frac{1}{n}\sum_{i=1}^{n} x_i$ est le maximum de la vraisemblance.

$$\boxed{\hat{\lambda}_{MV} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

Exemple 2

$X_i \sim N(m, \sigma^2)$ $\qquad \theta = \begin{pmatrix} m \\ \sigma^2 \end{pmatrix}$

Estimateur du max de vraisemblance de θ ?

$$\begin{cases} \frac{\partial \ln L}{\partial m} = 0 \\ \frac{\partial \ln L}{\partial \sigma^2} = 0 \end{cases} \iff \begin{cases} \frac{1}{\sigma^2}\left(\sum x_i - nm\right) = 0 \\ -\frac{n}{\cancel{2}\cancel{\sigma^2}} + \frac{1}{\cancel{2}\sigma^{\cancel{4}2}} \sum_{i=1}^{n} (x_i - m)^2 = 0 \end{cases}$$

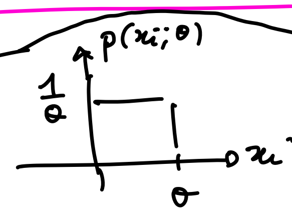
$$\iff \begin{cases} \boxed{m = \frac{1}{n}\sum_{i=1}^{n} x_i} \\ \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - m)^2 \end{cases}$$

L'estimateur du max de vraisemblance de $\theta = \begin{pmatrix} m \\ \sigma^2 \end{pmatrix}$ est

$$\hat{\theta}_{MV} = \begin{pmatrix} \hat{m}_{MV} \\ \hat{\sigma}^2_{MV} \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} x_i \\ \frac{1}{n}\sum_{i=1}^{n} (x_i - \hat{m}_{MV})^2 \end{pmatrix}$$

Exemple 3

$x_i \sim U(]0,\theta[)$



Estimateur du max de vraisemblance de θ

vraisemblance

$$L(x_1,\dots x_n;\theta) = \prod_{i=1}^{n} \frac{1}{\theta}\,\mathbb{1}_{]0,\theta[}(x_i)$$

$$= \frac{1}{\theta^n}\prod_{i=1}^{n} \mathbb{1}_{]0,\theta[}(x_i)$$

$$= \begin{cases} \dfrac{1}{\theta^n} & \text{si tous les } x_i \text{ sont } \le \theta \\ 0 & \text{sinon} \end{cases}$$

Log vraisemblance

$$\ln L = -n\ln\theta$$

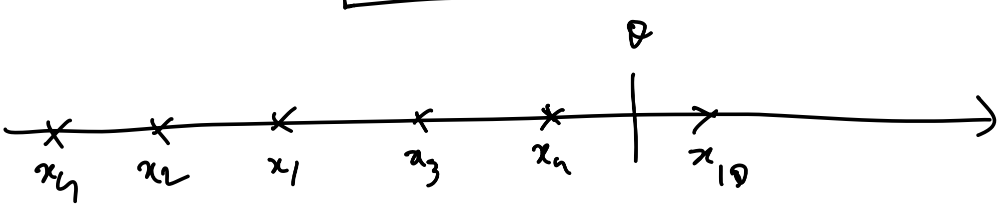
$$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} = 0$$

⚠ problème

BUG !!

$$L = \frac{1}{\theta^n}$$ fonction décroissante de θ

$$\boxed{x_i \le \theta}$$



$$\boxed{\hat{\theta}_{MV} = \underset{i=1}{\overset{n}{\mathrm{Max}}}\, X_i}$$

Méthode des moments

$X_i \sim \Gamma(a,b) \qquad \theta = \begin{pmatrix} a \\ b \end{pmatrix}$

estimateur des moments de θ

variance $= \dfrac{a}{b^2}$

$\downarrow$

$E[X^2] - E^2[X]$

$$\begin{cases} m_1 = E[X_i] = \dfrac{a}{b} \\ m_2 = E(X_i^2] = \dfrac{a}{b^2} + \left(\dfrac{a}{b}\right)^2 \end{cases}$$

$$\begin{cases} a = b\,m_1 \\ m_2 = \underbrace{\dfrac{b m_1}{b^2} + m_1^2}_{m_2 - m_1^2 = \frac{m_1}{b}} \end{cases} \iff \begin{cases} b = \dfrac{m_1}{m_2 - m_1^2} \\ a = \dfrac{m_1^2}{m_2 - m_1^2} \end{cases}$$

$$\hat{b}_{MO} = \frac{\hat{m}_1}{\hat{m}_2 - \hat{m}_1^2} = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i}{\frac{1}{n}\sum X_i^2 - \left(\frac{1}{n}\sum X_i\right)^2}$$

$$\hat{a}_{MO} = \frac{\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2}{\frac{1}{n}\sum X_i^2 - \left(\frac{1}{n}\sum X_i\right)^2}$$

COURS 6/11/2023

BAYES $$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Loi aposteriori du modèle

$$\boxed{p(\theta|x_1\dots x_n)} = \frac{\overbrace{p(x_1\dots x_n|\theta)}^{\text{vraisemblance}}\ \overbrace{p(\theta)}^{\text{loi a priori}}}{p(x_1\dots x_n)}$$

$$\propto p(x_1,\dots,x_n|\theta)\ p(\theta)$$

proportionnel

Estimateur MAP (du maximum a posteriori)

$$\boxed{\hat{\theta}_{MAP} = \underset{\theta}{\arg\max}\ p(\theta|x_1,\dots,x_n)}$$

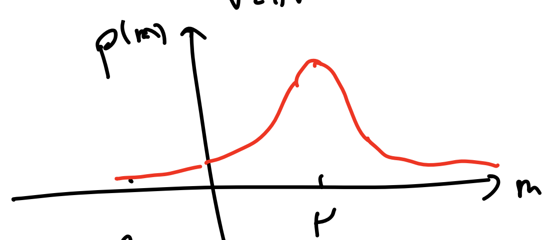
Exemple

$X_i \sim N(m, \sigma^2)$ — σ^2 connue

modèle statistique — $\theta = m$

$\theta = m \sim N(\mu, V^2)$

Loi a priori $$p(m) = \frac{1}{\sqrt{2\pi V^2}} \exp\left[-\frac{(m-\mu)^2}{2V^2}\right]$$



Estimateur MAP de m ?

vraisemblance

$$p(x_1,\dots,x_n\mid m) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i-m)^2}{2\sigma^2}\right]$$

$$= \boxed{\frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-m)^2\right)}$$

Loi a posteriori

$$p(m\mid x_1,\dots,x_n) \propto p(x_1,\dots,x_n\mid m)\times \boxed{p(m)}$$

$$\propto \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-m)^2 - \frac{1}{2v^2}(m-\mu)^2\right]$$

log de la loi a posteriori

$$\ln p(m\mid x_1,\dots,x_n) = K\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-m)^2\right) - \frac{1}{2v^2}(m-\mu)^2$$

$$\frac{\partial \ln p}{\partial m} = 0 \iff 0 - \frac{1}{2\sigma^2}\sum_{i=1}^{n} -2(x_i-m) - \frac{2}{2v^2}(m-\mu) = 0$$

$$\iff \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i - nm\right) - \frac{1}{v^2}(m-\mu) = 0$$

$$\iff \frac{1}{\sigma^2}\sum x_i + \frac{\mu}{v^2} = m\underbrace{\left[\frac{1}{v^2}+\frac{n}{\sigma^2}\right]}_{\frac{\sigma^2+nv^2}{v^2\sigma^2}}$$

donc $$m = \frac{v^2\sigma^2}{\sigma^2+nv^2}\left[\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i + \frac{\mu}{v^2}\right]$$

$$\boxed{\hat{m}_{MAP} = \frac{nv^2}{nv^2+\sigma^2}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{\sigma^2}{\sigma^2+nv^2}(\mu)}$$

Rappel $\hat{m}_{MV} = \frac{1}{n}\sum_{i=1}^{n} x_i$

quand $n \to +\infty$ (beaucoup de données)

$$\hat{m}_{MAP} \approx \frac{1}{n}\sum_{i=1}^{n} x_i$$

quand $n \to 0$ $\hat{m}_{MAP} \approx \mu$

Estimateur MMSE (minimum mean square error)

$$\boxed{\hat{\theta}_{MMSE} = E[\theta \mid x_1, \dots, x_n]}$$

Exemple

$X_i \sim N(m, \sigma^2)$ σ^2 connue $\theta = m$

$m \sim N(\mu, v^2)$

$\hat{m}_{MMSE}$?

On a

$$p(m \mid x_1, \dots, x_n) \propto \exp\Big[\underbrace{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - m)^2 - \frac{1}{2v^2}(m-\mu)^2}_{-(am^2+bm+c)}\Big]$$

$$\propto \exp\left[-\frac{(m-\alpha)^2}{2\beta^2}\right]$$

donc $\boxed{\begin{array}{l} m \mid x_1, \dots, x_n \sim \mathcal{N}(\alpha, \beta^2) \\ \text{avec } \alpha \text{ et } \beta^2 \text{ à déterminer} \end{array}}$

<u>coefficient de m^2</u>

$$\cancel{-}\frac{\cancel{1}\cancel{m^2}}{\cancel{2}\beta^2} = \cancel{-}\frac{1}{\cancel{2}\sigma^2} \times n\cancel{m^2} \cancel{-}\frac{1}{\cancel{2}v^2}\cancel{m^2}$$

$$\frac{1}{\beta^2} = \frac{n}{\sigma^2} + \frac{1}{v^2} = \frac{nv^2+\sigma^2}{\sigma^2 v^2}$$

donc $\boxed{\beta^2 = \frac{\sigma^2 v^2}{nv^2+\sigma^2}}$

<u>coefficient de m</u>

$$\cancel{-}\frac{1}{\cancel{2}\beta^2}(\cancel{-2}\alpha \cancel{m}) = \cancel{-}\frac{1}{\cancel{2}\sigma^2}(\cancel{-2}\cancel{m}\sum x_i) \cancel{-}\frac{1}{\cancel{2}v^2}(\cancel{-2}m\mu)$$

$$\frac{\alpha}{\beta^2} = \frac{1}{\sigma^2}\sum x_i + \frac{\mu}{v^2}$$

$$\alpha = \frac{\beta^2}{\sigma^2}\sum_{i=1}^n x_i + \mu\frac{\beta^2}{v^2}$$

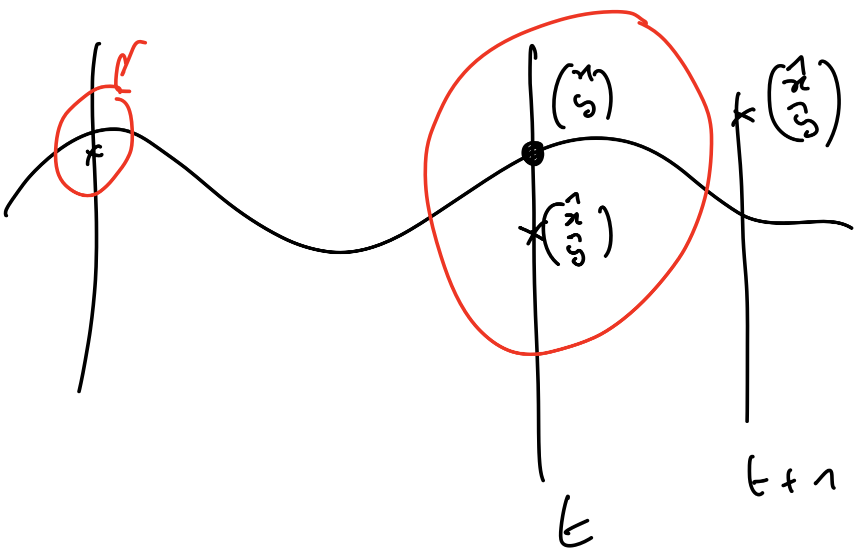
avec $\beta^2 = \frac{\sigma^2 v^2}{nv^2+\sigma^2}$

donc $\boxed{\alpha = \frac{v^2}{nv^2+\sigma^2}\sum_{i=1}^n x_i + \mu\frac{\sigma^2}{\sigma^2+nv^2}}$

<u>loi aposteriori</u> $\boxed{m \mid x_1 \dots x_n \sim \mathcal{N}(\alpha, \beta^2)}$

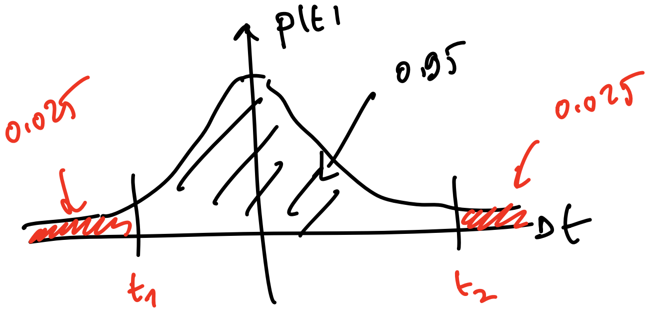
$$\hat{m}_{MMSE} = E[m \mid x_1, \dots, x_n] = \alpha$$

$$= \boxed{\frac{nv^2}{nv^2+\sigma^2}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{\mu\sigma^2}{\sigma^2+nv^2}}$$



Intervalle de confiance

$$T = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i - m}{\sigma/\sqrt{n}} \sim N(0,1)$$



F : fonction de répartition de la loi normale

$F(t_1) = 0.025$

$\boxed{t_1 = F^{-1}(0.025)}$

$\boxed{t_2 = F^{-1}(0.975) = -t_1}$

$F(t_2) = 0.975$

$$P[t_1 < T < t_2] = 0{,}95$$

$$T = \frac{\frac{1}{n}\sum x_i - m}{\sigma/\sqrt{n}}$$

$$P\left[\frac{\sigma}{\sqrt{n}} t_1 < \frac{1}{n}\sum x_i - m < t_2 \frac{\sigma}{\sqrt{n}}\right] = 0{,}95$$

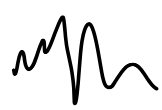
d'où

$$\boxed{P\left[\underbrace{\frac{1}{n}\sum x_i - \frac{t_2\sigma}{\sqrt{n}}}_{a} < m < \underbrace{\frac{1}{n}\sum x_i - \frac{\sigma}{\sqrt{n}} t_1}_{b}\right] = 0{,}95}$$

Cours du 9/11/2023

H_0 pas d'anomalie

H_1 anomalie



$\alpha = P[\text{Rejeter } H_0 \mid H_0 \text{ vraie}] = P[\text{on décide qu'il y a une anomalie} \mid \text{il n'y a pas d'anomalie}]$

$=$ PFA $=$ probabilité de fausse alarme

$\beta = P[\text{Rejeter } H_1 \mid H_1 \text{ vraie}] = P[\text{on décide qu'il n'y a pas d'anomalie} \mid \text{il y a une anomalie}]$

$=$ PND $=$ probabilité de non détection

Exemple de calculs de risques α et β et du seuil S_α

H_0 $m = m_0$

H_1 $m = m_1$ $(> m_0)$

$X_i \sim N(m, \sigma^2)$
σ^2 connue

Rejet de H_0 si $T = \frac{1}{n} \sum_{i=1}^{n} x_i > S_\alpha$

$\alpha = P[\text{Rejeter } H_0 \mid H_0 \text{ vraie}]$

$= P[T > S_\alpha \mid m = m_0]$

$x_i \sim N(m_0, \sigma^2)$

$$\alpha = 1 - P\Big[\underbrace{\frac{1}{n}\sum_{i=1} x_i}_{} \leq S_\alpha \;\Big|\; x_i \sim N(m_0, \sigma^2)\Big]$$

$$\underbrace{\left(\tfrac{1}{n} \cdots \tfrac{1}{n}\right)}_{}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \sim N\left(m_0, \frac{\sigma^2}{n}\right)$$

$E\left[\frac{1}{n}\sum x_i\right] = \frac{1}{n}\sum_{i=1}^{n} \underbrace{E(x_i)}_{m_0}$

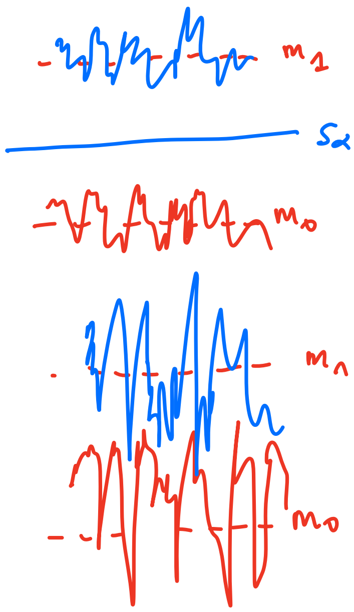
$Var\left[\frac{1}{n}\sum x_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} \underbrace{var\, x_i}_{\sigma^2}$

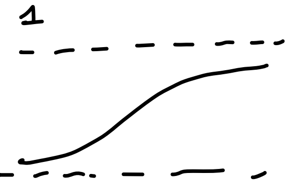
$$\alpha = 1 - P\left[\underbrace{\frac{\frac{1}{n}\sum_{i=1}^{n} x_i - m_0}{\sqrt{\frac{\sigma^2}{n}}}}_{U} \leq \frac{S_\alpha - m_0}{\sqrt{\sigma^2/n}} \;\middle|\; U \sim N(0,1)\right]$$

$$\boxed{\alpha = 1 - F_{N(0,1)}\left(\frac{S_\alpha - m_0}{\sqrt{\sigma^2/n}}\right)}$$

Recherche du seuil $\quad 1 - \alpha = F_{N(0,1)}\left(\frac{S_\alpha - m_0}{\sqrt{\frac{\sigma^2}{n}}}\right)$

$$F^{-1}_{N(0,1)}(1-\alpha) = \frac{S_\alpha - m_0}{\sigma/\sqrt{n}}$$





$$\boxed{S_\alpha = m_0 + \underbrace{\frac{\sigma}{\sqrt{n}} F^{-1}_{N(0,1)}(1-\alpha)}}$$

<u>Remarque</u> : on rejette H_0 si $\frac{1}{n}\sum_{i=1}^{n} x_i > S_\alpha$

$\underline{\alpha = 0} = P[\text{Rejeter } H_0 \mid H_0 \text{ vraie}]$ //

$$S_0 = m_0 + \frac{\sigma}{\sqrt{n}} \times +\infty = +\infty$$

on ne rejette jamais H_0

$\underline{\alpha = 1}$: $S_1 = m_0 + \frac{\sigma}{\sqrt{n}} F^{-1}_{N(0,1)}(0) = -\infty$

on rejette toujours H_0

$\boxed{\text{Calcul du risque } \beta \text{ ou de } \Pi = 1-\beta}$

$\beta = P[\underbrace{\text{Rejeter } H_1}_{\text{accepter } H_0} \mid \overset{m = m_1}{H_1 \text{ vraie}}]$

$\Pi = P[\text{Rejeter } H_0 \mid H_1 \text{ vraie}]$

$\alpha = P[\text{Rejeter } H_0 \mid \overset{m = m_0}{H_0 \text{ vraie}}]$

$$\Pi = P\left[\frac{1}{n}\sum_{i=1}^{n} x_i > S_\alpha \mid m = m_1\right]$$

$= \ldots$

donc $\boxed{\Pi = 1 - F_{N(0,1)}\left(\frac{S_\alpha - m_1}{\sigma/\sqrt{n}}\right)}$

$$\boxed{\beta = 1 - \Pi = F_{N(0,1)}\left(\frac{S_\alpha - m_1}{\sigma/\sqrt{n}}\right)}$$

$$\boxed{S_\alpha = m_0 + \frac{\sigma}{\sqrt{n}} F^{-1}_{N(0,1)}(1-\alpha)}$$

<u>Courbe COR</u>

$$\boxed{\Pi = 1 - F_{N(0,1)}\left[\frac{m_0 - m_1}{\sigma/\sqrt{n}} + F^{-1}_{N(0,1)}(1-\alpha)\right]}$$

Π en fonction de α

$$\frac{\sqrt{n}\,(m_0 - m_1)}{\sigma} = -\frac{\sqrt{n}}{\sigma}(m_1 - m_0)$$

Test de NEYMAN PEARSON

$\boxed{H_0}$ $m = m_0$

$\boxed{H_1}$ $m = m_1$

$X_i \sim N(m, \sigma^2)$ connue

NP ?

Rejet de H_0 si
$$\frac{\prod_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left[-\frac{(x_i - m_1)^2}{2\sigma^2}\right]}{\prod_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left[-\frac{(x_i - m_0)^2}{2\sigma^2}\right]} > \text{Seuil} = S_{1,\alpha}$$

Un test équivalent rejette H_0 si
$$\sum_{i=1}^{n}\left[-\frac{1}{2\sigma^2}(x_i - m_1)^2 + \frac{1}{2\sigma^2}(x_i - m_0)^2\right] > S_{2,\alpha} \quad //$$

$$+\frac{1}{2\sigma^2}\sum_{i=1}^{n} -\left(-2m_1 x_i + m_1^2\right) + \left(-2m_0 x_i + m_0^2\right) > S_{2,\alpha}$$

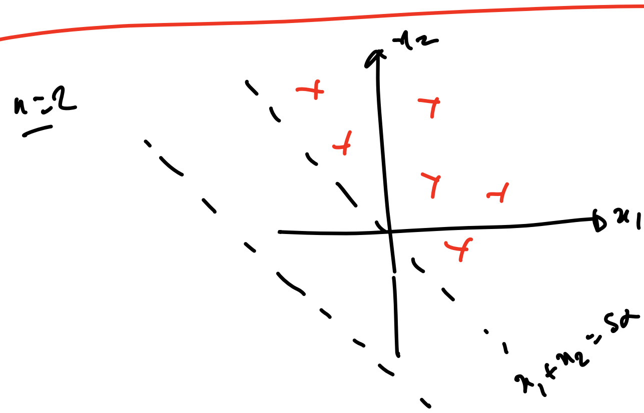
$$2(m_1 - m_0)\sum_{i=1}^{n} x_i > \underbrace{2\sigma^2 S_{2,\alpha} + nm_1^2 - nm_0^2}_{S_{3,\alpha}}$$

$m_1 > m_0$ donc $m_1 - m_0 > 0$ d'où

$$\boxed{\text{Rejet de } H_0 \text{ si } \sum_{i=1}^{n} x_i > S_\alpha}$$

Statistique de test $T = \sum_{i=1}^{n} X_i$

Zone de rejet de H_0 = région critique du test = $\left\{(x_1,\dots,x_n) \,/\, \sum_{i=1}^{n} x_i > S_\alpha \right\}$



TD du 8/11/2023

Loi de Weibull $f(x;\theta,\lambda) = \frac{\lambda}{\theta} x^{\lambda-1} \exp\left(-\frac{x^\lambda}{\theta}\right) \quad x>0$

λ connu donc on estime θ

1) Loi de $U = X^\lambda$, $E(U)$, $Var\, U$

2) $\hat{\theta}_{MV}$? estimateur sans biais ?, convergent ? efficace ? Erreur quadratique moyenne ?

Réponses $\boxed{U = X^\lambda \Rightarrow X = U^{1/\lambda}}$

→ densité de U $\quad \pi(u) = \frac{\lambda}{\theta}\left(u^{1/\lambda}\right)^{\lambda-1} \exp\left(-\frac{u}{\theta}\right) \overbrace{\left|\frac{dx}{du}\right|}^{\text{Jacobien}}$

$x = u^{1/\lambda} \Rightarrow \frac{dx}{du} = \frac{1}{\lambda} u^{\frac{1}{\lambda}-1}$

donc $\pi(u) = \frac{\cancel{\lambda}}{\theta} u^{\cancel{\frac{\lambda-1}{\lambda}}} \exp\left(-\frac{u}{\theta}\right) \cancel{\frac{1}{\lambda}} u^{\cancel{\frac{1}{\lambda}-1}} \quad u>0$

(le changement de variable $U = X^\lambda$ est bijectif de $\mathbb{R}^+$ dans $\mathbb{R}^+$)

donc $\boxed{\pi(u) = \frac{1}{\theta}\exp\left(-\frac{u}{\theta}\right) \quad u>0}$

donc $U \sim G\left(\frac{1}{\theta}, 1\right)$

Table

$$\boxed{E[U] = \theta \quad Var\, U = \theta^2}$$

2) $$L(x_1, \dots, x_n ; \theta) = \prod_{i=1}^{n} \left(\frac{d}{\theta}\right) x_i^{d-1} \exp\left(-\frac{x_i^d}{\theta}\right)$$

$$\propto \prod_{i=1}^{n} \frac{1}{\theta} \exp\left(-\frac{x_i^d}{\theta}\right) = \frac{1}{\theta^n} \exp\left(-\frac{\sum x_i^d}{\theta}\right)$$

proportionel

$$\frac{\partial \ln L}{\partial \theta} \geq 0 \iff \frac{\partial}{\partial \theta}\left[-n \ln \theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i^d\right] \geq 0$$

$$\iff -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i^d \geq 0$$

θ	0		$\frac{1}{n}\sum x_i^d$		$+\infty$
$\frac{\partial \ln L}{\partial \theta}$		$+$	0	$-$	
		$\nearrow$		$\searrow$	

on $\otimes$ tout par θ^2 $\Rightarrow$ $$\boxed{\theta \leq \frac{1}{n}\sum_{i=1}^{n} x_i^d}$$

donc $$\boxed{\hat{\theta}_{MV} = \frac{1}{n}\sum_{i=1}^{n} X_i^d}$$

<u>Biais</u> $$E[\hat{\theta}_{MV}] = \frac{1}{n}\sum_{i=1}^{n} \underbrace{E[X_i^d]}_{\theta} = \frac{n\theta}{n} = \theta$$

donc $\boxed{b_n(\theta) = 0}$

$$Var(\hat{\theta}_{MV}) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i^d\right) \underset{X_1 \dots X_n \text{ ind}}{=} \frac{\sum_{i=1}^{n} \overbrace{Var(X_i^d)}^{\theta^2}}{n^2} = \boxed{\frac{\theta^2}{n}}$$

Biais $= 0$

$Var(\hat{\theta}_{MV}) \xrightarrow[n \to +\infty]{} 0$

donc $\boxed{\hat{\theta}_{MV} \text{ estimateur convergent de } \theta}$

<u>Efficacité</u> : on a $$\frac{\partial \ln L}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i^d$$

donc $$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i^d$$

$$E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} \underbrace{E[X_i^d]}_{\theta}$$

$$= \frac{n}{\theta^2} - \frac{2n\theta}{\theta^3} = \frac{n}{\theta^2} - \frac{2n}{\theta^2} = \frac{-n}{\theta^2}$$

$$\boxed{BCR = \frac{\theta^2}{n}}$$

$\mathrm{Var}\,\hat{\theta} = \frac{\theta^2}{n} = BCR$

$\hat{\theta}_{MV}$ estimateur sans biais $\Rightarrow$ $\boxed{\hat{\theta}_{MV} \text{ est l'estimateur efficace de } \theta}$

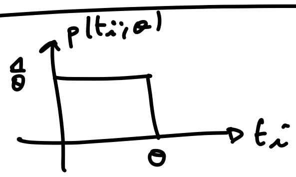
Erreur quadratique moyenne

$$E[(\hat{\theta} - \theta)^2] = (\text{biais})^2 + \text{Variance}$$
$$= 0^2 + \frac{\theta^2}{n} = \boxed{\frac{\theta^2}{n}}$$

Exo 1

$T_i \sim U(]0,\theta[)$

1)



$$\begin{cases} E(T_i) = \frac{\theta}{2} \\ \mathrm{Var}(T_i) = \frac{\theta^2}{12} \end{cases}$$ Tables

2) $\bar{T} = \frac{1}{n}\sum_{i=1}^{n} T_i$

- moyenne et variance de $\bar{T}$
- en déduire que $\hat{\theta}_1 = 2\bar{T}$ est un estimateur sans biais et convergent de θ

$$E[\bar{T}] = \frac{1}{n}\sum_{i=1}^{n} \underbrace{E(T_i)}_{\frac{\theta}{2}} = \frac{1}{n}\frac{n\theta}{2} = \boxed{\frac{\theta}{2}} \neq \theta$$

donc $\boxed{\bar{T} \text{ estimateur biaisé de } \theta}$

$$\mathrm{Var}(\bar{T}) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} T_i\right) \underset{T_1,\dots,T_n \text{ ind}}{=} \frac{1}{n^2}\sum_{i=1}^{n} \underbrace{\mathrm{Var}\,T_i}_{\frac{\theta^2}{12}} = \boxed{\frac{\theta^2}{12n}}$$

$\hat{\theta}_1 = 2\bar{T}$ est tel que $E(\hat{\theta}_1) = 2E(\bar{T}) = \boxed{\theta}$

$\mathrm{Var}(\hat{\theta}_1) = 4\,\mathrm{Var}\,\bar{T} = 4 \times \frac{\theta^2}{12n} = \boxed{\frac{\theta^2}{3n}}$

donc $\hat{\theta}_1$ est un estimateur sans biais et convergent de θ !

Rq : $\hat{\theta}_1$ est un estimateur des moments de θ

3) Estimateur du max de vraisemblance

$$L(t_1,\dots,t_n;\theta) = \prod_{i=1}^{n}\left[\frac{1}{\theta}\mathbb{1}_{]0,\theta[}(t_i)\right]$$

$$= \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}_{]0,\theta[}(t_i)$$

$$= \begin{cases} \frac{1}{\theta^n} & \text{si tous les } t_i \text{ sont } \le \theta \\ 0 & \text{sinon} \end{cases}$$

$$L(t_1,\dots,t_n;\theta) = \underbrace{\frac{1}{\theta^n}}_{\text{fonction } \searrow \text{ de } \theta} \times \underbrace{\prod_{i=1}^{n} I_{]0,\theta[}(t_i)}_{=1 \text{ si tous les } t_i \text{ sont } \le \theta}$$

i.e si θ est $>$ à tous les t_i

donc θ qui maximise $L(t_1,\dots,t_n;\theta)$ est $\theta = \underset{i=1,\dots,n}{\text{Max}}\, t_i$

$$\boxed{\hat{\theta}_{MV} = \underset{i=1,\dots,n}{\text{Max}}\, T_i}$$

or $E(\hat{\theta}_{MV})$?

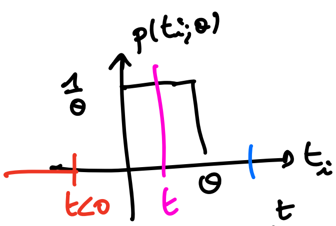
$Var(\hat{\theta}_{MV})$?

$\boxed{\text{On doit trouver la loi de } \underset{i}{\text{Max}}\, T_i = Y_n}$

Fonction de répartition de Y_n : $P[Y_n < t] = P[\underset{i}{\text{Max}}\, T_i < t]$

$= P[T_1 < t, T_2 < t, \dots, T_n < t]$

$= \prod_{i=1}^{n} P[T_i < t]$



$$P[T_i < t] = \int_{-\infty}^{t} p(t_i;\theta)\,dt_i = \begin{cases} 0 & t<0 \\ 1 & t>\theta \\ \underbrace{\int_0^t \frac{1}{\theta}\,dt_i}_{\frac{t}{\theta}} & t \in]0,\theta[\end{cases} = \begin{cases} 0 & t<0 \\ 1 & t>\theta \\ \frac{t}{\theta} & t\in]0,\theta[\end{cases}$$

$$P[Y_n < t] = \prod_{i=1}^{n} \left(\frac{t}{\theta}\right) \qquad t \in]0,\theta[$$

$$= \frac{t^n}{\theta^n}$$

La densité de Y_n est $\boxed{p(t) = \frac{1}{\theta^n}\, n t^{n-1} \qquad t\in]0,\theta[}$

$$E[Y_n] = \int t\, p(t)\,dt = \int_0 \frac{1}{\theta^n}\, n t^{n-1}\, t\, dt$$

$$= \frac{1}{\theta^n} n \left[\frac{t^{n+1}}{n+1}\right]_0^\theta$$

$$\boxed{E[Y_n] = \frac{n}{n+1}\,\theta}$$

Un estimateur non biaisé de θ est

$$\boxed{\hat\theta_2 = \frac{n+1}{n} Y_n = \frac{n+1}{n} \operatorname{Max}_{i=1}^{n} \{T_i\}}$$

Pour déterminer le meilleur estimateur, il faut comparer

$\operatorname{Var}\hat\theta_1 = \frac{\theta^2}{3n}$ et $\operatorname{Var}\hat\theta_2$!

Calcul de la variance

$$\operatorname{Var}\hat\theta_2 = \operatorname{Var}\left(\frac{n+1}{n} Y_n\right) = \left(\frac{n+1}{n}\right)^2 \underbrace{\operatorname{Var}(Y_n)}_{E(Y_n^2) - E^2[Y_n]}$$

on a calculé $E[Y_n] = \frac{n}{n+1}\theta$

$$E[Y_n^2] = \int t^2 p(t)\,dt = \int_0^\theta t^2 \frac{1}{\theta^n} n t^{n-1}\, dt$$

$$= \frac{1}{\theta^n} n \left[\frac{t^{n+2}}{n+2}\right]_0^\theta$$

$$= \frac{n}{n+2}\theta^2$$

$$\text{donc } \operatorname{var}(Y_n) = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2$$

$$= \frac{n\theta^2}{(n+2)(n+1)^2}\left[\underbrace{(n+1)^2 - n(n+2)}_{n^2+2n+1\ -n^2-2n}\right]$$

$$\operatorname{var}(Y_n) = \frac{n\theta^2}{(n+2)(n+1)^2}$$

$$\text{d'où } \operatorname{Var}(\hat\theta_2) = \frac{\cancel{(n+1)^2}}{\cancel{n^2}} \frac{\cancel{n}\theta^2}{(n+2)\cancel{(n+1)^2}} \boxed{= \frac{\theta^2}{n(n+2)}} \sim \boxed{\frac{\theta^2}{n^2}}$$