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On the statistics of estimated reflection and cepstrum coefficients of an autoregressive process

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Abstract

A common assumption in most theoretical and applied works is the Gaussian assumption. Many signal processing tools have been developed using this Gaussian hypothesis. But in recent years, non-Gaussian processes have been considered with increasing interest. New techniques, such as those based on higher order statistics, have been studied for these processes. Thus, the Gaussian or non-Gaussian nature of signals, parameters or estimators has to be indicated. The aim of this paper is to study the statistical properties of estimated reflection and cepstrum coefficients. First, a recursive way of computing their probability density function (p.d.f.) from that of estimated AR parameters is proposed. Then, these probability density functions are compared to the Gaussian law. The last part of the paper is developed to the study of the convexity of the p.d.f. Through examples, they are shown to be possibly non-convex, yielding a case in which the centroid distance classifier may result in a failure.

Zusammenfassung

In den meisten theoretischen und angewandten Arbeiten benutzt man die Annahme nach Gauss. Basierend auf dieser Hypothese sind zahlreiche Signalverarbeitungshilfsmittel entwickelt worden. In den letzten Jahren allerdings wurden nicht-Gauss'sche Prozesse mit wachsendem Interesse in Betracht gezogen. Neue Techniken, wie solche, die auf höherer statistischer Ordnung basieren, wurden für diese Prozesse entwickelt. Die Gauss'sche oder nicht-Gauss'sche Eigenschaft von Signalen, Parametern oder Estimatoren muss deshalb präzisiert werden. In diesem Artikel untersuchen wir die statistischen Eigenschaften von Cepstrum- und Reflexionskoeffizienten, zwei äquivalenten Darstellungen für AR-Parameter. Wir beschreiben zunächst einen rekursiven Algorithmus zur Berechnung ihrer Wahrscheinlichkeitsdichten als Funktion der Wahrscheinlichkeitsdichte der AR parameter. Anschliessend vergleichen wir die erhaltenen Wahrscheinlichkeitsdichten mit der Gauss'schen Wahrscheinlichkeitsdichte. Schliesslich geben wir einige Beispiele, die zeigen, dass die gewonnenen Wahrscheinlichkeitsdichten nicht immer konvex sind, was zu ungewünschten Resultaten bei der Klassifizierung unter Anwendung des Schwerpunktkriteriums führen kann.

Résumé

Une hypothèse habituelle rencontrée dans la plupart des travaux théoriques et appliqués est l'hypothèse gaussienne. Beaucoup d'études en traitement du signal ont été effectuées sous cette hypothèse. Mais durant ces dernières années, dans un souci d'une représentation plus proche de la réalité, les processus non gaussiens ont été considérés avec beaucoup

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d'intérêt. De nouvelles techniques, comme celles basées sur les statistiques d'ordre supérieur, ont été mises au point pour ces processus. La nature gaussienne ou non gaussienne des signaux, paramètres ou estimateurs doit donc être précisée. Le but de cet article est d'étudier les propriétés statistiques des estimations de deux ensembles de coefficients équivalents aux paramètres AutoRegressifs (AR): les coefficients de réflexion et les coefficients cepstraux. Dans un premier temps, nous présentons une méthode récursive permettant de déterminer la densité de probabilité théorique des estimations de ces coefficients en fonction de celle des estimations des paramètres AR. Dans un second temps, nous comparons cette densité de probabilité à la loi normale. Dans une dernière partie, nous montrons, à l'aide de quelques exemples, que cette densité de probabilité peut être non convexe, ce qui permet d'expliquer certains mauvais résultats obtenus lors d'une classification avec ces coefficients et la règle de la distance au barycentre.

Keywords: Non-Gaussian; Parametric modelling; Cepstrum coefficients; Reflection coefficients; Classification

1. Introduction

Autoregressive (AR) parametric modeling is successfully used in many signal processing fields such as spectral analysis or classification. But, in many applications, several equivalent parameters, reflection and cepstrum coefficients among these, are preferred to AR parameters. For instance, several authors have emphasized the noteworthy results obtained in pattern recognition with cepstrum coefficients [4, 11] and reflection coefficients have desirable properties for quantization and data compression [7, 9]. The aim of this paper is to study the reflection and cepstrum coefficient probability density function (p.d.f.).

These two sets of parameters are generally computed from AR parameters. *The first step in our study is to propose a recursive way of computing the exact probability density function of these two estimated parameter sets from that of estimated AR parameters.* There are numerous applications for which the estimated AR parameter statistics is assumed to be known and for which the estimated reflection or cepstrum coefficients are of great interest. For instance:

- In pattern recognition, AR parameters are random and their statistics which characterises within-class scattering is usually assumed to be known (generally, Gaussian). Knowledge of reflection or cepstrum coefficients allows us to determine the shape of point clusters obtained when classifying with Bayesian techniques using these parameters [2, 14, 15].
- In estimation theory, AR parameters are deterministic and have to be estimated. According to the Mann and Wald theorem [10], most commonly used AR parameter estimators may be assumed to be Gaussian when these parameters are estimated for sufficiently large data records [7]. The estimated reflection or cepstrum coefficient statistics may then be computed. It is of great interest while designing for optimum quantization [7] or coding [5].

In a second step, estimated AR parameters are assumed to be Gaussian. The estimated reflection and cepstrum coefficient statistics are studied, with respect to their Gaussian or non-Gaussian behaviour.

The last part of this paper is devoted to the study of estimated reflection and cepstrum coefficient p.d.f. convexity. It is well known that surprising results may be obtained when classifying with parameters the p.d.f. of which is not convex [14, 15]. In the case of Gaussian AR parameters, estimated reflection and cepstrum coefficient p..d.f. are shown to be possibly non-convex, yielding a case in which the centroid distance classifier may result in a failure.

2. Reflection coefficients

A usual way of estimating the reflection coefficients k_i of an AR process is to estimate AR parameters with a least squares technique and then to use the following relations:

$$i = 1, \ldots, p, \quad \hat{k}_i = \hat{a}_i^{(i)}, \tag{1}$$

$$1 \leqslant j \leqslant i-1, \quad \hat{a}_j^{(i-1)} = \frac{\hat{a}_j^{(i)} - \hat{a}_j^{(i)} \hat{a}_{i-j}^{(i)}}{1 - \hat{k}_i^2}. \tag{2}$$

The parameters $\hat{a}_j^{(i)}$, $j = 1, \ldots, p$ are the ith order linear predictor estimated coefficients [6]. For $i = p$, these parameters are confounded with AR parameters:

$$\hat{a}_j^{(p)} = \hat{a}_j, \quad j = 1, \ldots, p.$$

Vectors $[\hat{a}_1^{(i)}, \ldots, \hat{a}_p^{(i)}]$ are computed recursively for $i = p - 1, \ldots, 1$ from AR parameter values and relations (1) and (2). Each step allows us to estimate one reflection coefficient $\hat{k}_i = \hat{a}_i^{(i)}$.

2.1. Estimated reflection coefficient statistics

The aim of this part of the paper is to present a recursive means of determining the estimated reflection coefficient p.d.f. from that of AR parameters. Let us consider the following vector $\hat{V}_i$:

$$\hat{V}_i = (\hat{a}_p^{(p)}, \hat{a}_{p-1}^{(p-1)}, \ldots, \hat{a}_i^{(i)}, \hat{a}_{i-1}^{(i)} \hat{a}_{i-2}^{(i)}, \ldots, \hat{a}_1^{(i)})^{\mathrm{T}},$$
$$i = 1, \ldots, p.$$

Note that this vector may be split into two:

- the first one contains $p - i + 1$ components equal to the $p - i + 1$ last reflection coefficients $\hat{k}_p, \ldots, \hat{k}_i$:

$$\begin{bmatrix} \hat{a}_p^{(p)} \\ \vdots \\ \hat{a}_i^{(i)} \end{bmatrix} = \begin{bmatrix} \hat{k}_p \\ \vdots \\ \hat{k}_i \end{bmatrix};$$

- the second one has $i - 1$ components:

$$\begin{bmatrix} \hat{a}_{i-1}^{(i)} \\ \vdots \\ \hat{a}_1^{(i)} \end{bmatrix}.$$

In particular, for $i = p$, $\hat{V}_p$ is the estimated AR parameter vector $\hat{V}_p = [\hat{a}_p, \hat{a}_{p-1}, \ldots, \hat{a}_1]^{\mathrm{T}}$, the p.d.f. of which is assumed to be known and for $i = 1$, $\hat{V}_1 = [\hat{k}_p, \hat{k}_{p-1}, \ldots, \hat{k}_1]^{\mathrm{T}}$, the p.d.f. of which is under study.

The next step in our study is to determine the $\hat{V}_{i-1}$ p.d.f. as a function of that of $\hat{V}_i$. There is a one-to-one transformation between $\hat{V}_i$ and $\hat{V}_{i-1}$ components:

- the first $p - i + 1$ $\hat{V}_i$ and $\hat{V}_{i-1}$ components are equal;
- the last $i - 1$ $\hat{V}_{i-1}$ components, that is to say $[\hat{a}_{i-1}^{(i-1)}, \ldots, \hat{a}_1^{(i-1)}]$, are linked to those of $\hat{V}_i$ by the following relations:

$$1 \leqslant j \leqslant i - 1, \quad \hat{a}_j^{(i-1)} = \frac{\hat{a}_j^{(i)} - \hat{a}_i^{(i)} \hat{a}_{i-j}^{(i)}}{1 - \hat{a}_i^{(i)2}}. \tag{3}$$

Inverting relation (3) allows us to determine the Jacobian matrix of the transformation between $\hat{V}_i$ and $\hat{V}_{i-1}$ (see Appendix A). This allows us to determine the $\hat{V}_{i-1}$ p.d.f. (denoted by $f_{i-1}(x_p, \ldots, x_1)$) as a function of that of $\hat{V}_i$ denoted by $f_i(x_p, \ldots, x_1)$. The following results are obtained:

(i) when i is odd,

$$f_{i-1}(x_p, \ldots, x_1) = (1 - x_i^2)^{(i-1)/2}$$
$$\times f_i(x_p, \ldots, x_i, x_{i-1} + x_i x_1, \ldots, x_1 + x_i x_{i-1}), \tag{4}$$

(ii) when i is even,

$$f_{i-1}(x_p, \ldots, x_1)$$
$$= (1 + x_i)(1 - x_i^2)^{(i-2)/2}$$
$$\times f_i(x_p, \ldots, x_i, x_{i-1} + x_i x_1, \ldots,$$
$$(1 + x_i) x_{i/2}, \ldots, x_1 + x_i x_{i-1}). \tag{5}$$

Assuming that the AR parameter vector $\hat{V}_p = [\hat{a}_p, \ldots, \hat{a}_{p-1}, \ldots, \hat{a}_1]^{\mathrm{T}}$ p.d.f. is known, that of $\hat{V}_1 = [\hat{a}_p^{(p)}, \ldots, \hat{a}_1^{(1)}]^{\mathrm{T}} = [\hat{k}_p, \ldots, \hat{k}_1]^{\mathrm{T}}$ can then be determined recursively. The $\hat{k}_i$ statistics can be computed with numerical integration with respect to variables $(\hat{k}_p, \ldots, \hat{k}_{i+1}, \hat{k}_{i-1}, \ldots, \hat{k}_1)$. For high orders, this numerical integration can be performed using Monte Carlo methods [12] or genetic programming [8].

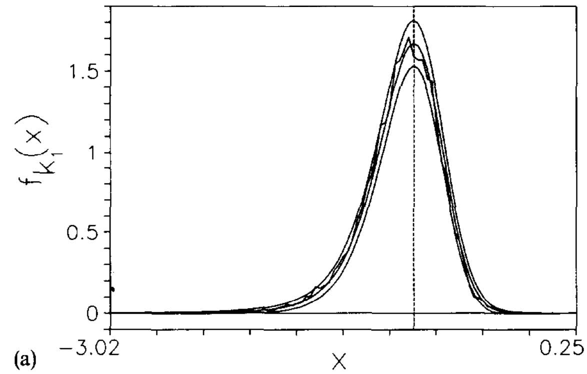
2.2. The Gaussian case

There are a lot of applications for which the estimated AR parameter distribution is assumed to be Gaussian. In estimation theory, this hypothesis is based on Mann and Wald theorem when AR parameters are estimated for sufficiently large data records [10]. In pattern recognition, AR parameter statistics which characterises within-class scattering is usually assumed to be Gaussian [2]. The aim of this part is to study the estimated reflection parameter p.d.f. in the case of Gaussian estimated AR parameters. *AR parameters and reflection coefficients, being non-linearly transformed, cannot both be Gaussian.* To illustrate this property, let us consider an order 2 Gaussian vector $\hat{a} = [\hat{a}_1, \hat{a}_2]$ such that $E[\hat{a}] = a = [a_1, a_2]$. The form of the AR parameter covariance matrix is linked to the physical phenomena studied. For instance, in spectral analysis, when AR parameters are estimated with the autocorrelation method of linear prediction [6], it is of the form $\Sigma_{\hat{a}} = (\sigma_v^2/N)R_x^{-1}$, σ_v^2, N and R_x being respectively the input noise variance, the signal point number and the AR model output covariance matrix. In this particular case, the inverse of the AR model output covariance matrix denoted by R_x^{-1} may be determined as a function of the AR parameter vector [3]. This particular case can raise some problems because Gaussian AR parameter estimations can be such that corresponding poles lie outside the unit circle. Indeed, this situation cannot occur when AR parameters are estimated with the autocorrelation method of linear prediction. That is why the pattern recognition case has been preferred to the estimation one. In the former, the AR parameter covariance matrix can be of any kind and can lead to non-minimum phase AR parameters lying outside the unit circle. For our simulations, the circular Gaussian case defined by $\Sigma_{\hat{a}} = \sigma^2 I$, I being the identity matrix, has been considered. The covariance matrix is chosen proportional to the identity matrix in order to simplify expressions (4) and (5) but other cases have been studied and give similar results [13]. In the case of two AR poles,

$$p_1 = 0.8\exp[\mathrm{j}\,\tfrac{\pi}{4}], \qquad p_2 = 0.8\exp[-\mathrm{j}\,\tfrac{\pi}{4}],$$

with $\sigma^2 = 0.07$, the results obtained are shown in Fig. 1.

As can be seen above, experimental and theoretical results are in good agreement. The first estimated reflection coefficient is clearly non-Gaussian (asymmetric p.d.f.). The last one, which is equal to the last AR parameter, is Gaussian, as Fig. 1(b) confirms. However, in estimation theory, when AR parameters are estimated with the autocorrelation method of linear prediction [6], estimated reflection coefficients are asymptotically Gaussian [7]. In a similar way, in pattern recognition, the estimated reflection coefficient p.d.f. tends to the Gaussian law when parameter σ^2 tends to zero. In the next part, some distances between estimated reflection p.d.f. and the Gaussian one are studied as a function of the parameter σ^2. The aim of this part is to illustrate through two particular models, the well-known asymptotic normality of estimated reflection coefficient p.d.f.



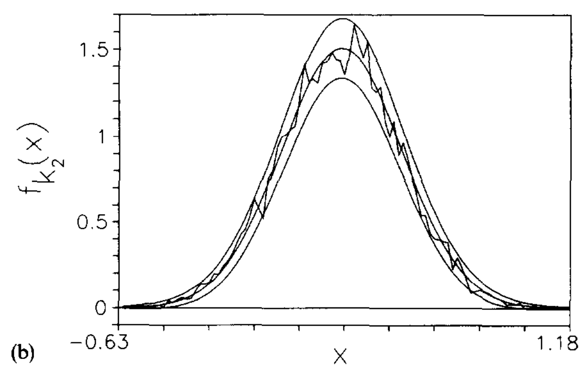


Fig. 1. Estimated and theoretical reflection coefficient p.d.f. with 95% confidence intervals: (a) coefficient $\hat{k}_1$, (b) coefficient $\hat{k}_2$.

2.3. Distance to the Gaussian law

Two models denoted by *Model 1* and *Model 2* are considered. The first one has two conjugated complex poles close to the unit circle:

$$p_1 = 0.9\exp[\mathrm{j}\tfrac{\pi}{4}], \qquad p_2 = 0.9\exp[-\mathrm{j}\tfrac{\pi}{4}].$$

The second model has two conjugated complex poles close to the origin:

$$p_1 = 0.2\exp[\mathrm{j}\tfrac{\pi}{4}], \qquad p_2 = 0.2\exp[-\mathrm{j}\tfrac{\pi}{4}].$$

For these two models, a Gaussian vector with mean $E[\hat{a}] = a = [a_1, a_2]^{\mathrm{T}}$ and covariance matrix $\Sigma_{\hat{a}} = \sigma^2 I$ has been generated. Different σ^2 parameter values have been considered ($\sigma^2 = 10, 5, 1, 0.5, 0.1, 5\times10^{-2}, 1\times10^{-2}, 5\times10^{-3}, 1\times10^{-3}$) and for each σ^2 value, reflection coefficients have been estimated from AR parameter estimations using relations (1) and (2). Estimated reflection coefficient p.d.f. has then been determined from Eqs. (4) and (5) and estimated with a histogram. It has then been compared to the normal distribution with three distances: the Kolmogorov distance, the Kullback divergence and the Bhattacharyya distance [1]. The Kolmogorov distance between the estimated X variable p.d.f. denoted by f_x and the Gaussian one f is defined by

$$d_1(f_X, f) = \sup_{t\in\mathbb{R}} |\hat{F}_X(t) - \Phi(t)|,$$

$\hat{F}_X(t)$ being the X variable cumulative histogram computed from N observations and $\Phi(t) = \int_{-\infty}^{t}(1/\sqrt{2\pi})\exp[-\frac{u^2}{2}]\,\mathrm{d}u$ being the normal distribution function.

The Kullback divergence and the Bhattacharyya distance between the two p.d.f. f_1 and f_2, denoted respectively by d_2 and d_3, are defined by

$$d_2(f_1, f_2) = \int_{\mathbb{R}} [f_2(x) - f_1(x)] \ln\frac{f_2(x)}{f_1(x)}\,\mathrm{d}x,$$

$$d_3(f_1, f_2) = -\ln\left[\int_{\mathbb{R}} \sqrt{f_1(x) f_2(x)}\,\mathrm{d}x\right].$$

Note that the first distance is a distance between an estimated p.d.f. (histogram) and the exact p.d.f. of the Gaussian distribution whereas the other two are distances between two exact p.d.f. The second estimated reflection coefficient, which is equal to the second AR parameter (1), is Gaussian. Consequently, only the first one has been considered. The first distance has to be computed for a high signal point number N: the higher the value of N, the closer the estimated reflection p.d.f. to the exact one. For $N = 10{,}000$, the results obtained are shown in Fig. 2.

In each figure is indicated the 5% Kolmogorov bound under which the Gaussian hypothesis may be accepted in a Kolmogorov test. These simulations show that the distance between the first estimated reflection coefficient statistics and the Gaussian one decreases with parameter σ^2 and becomes lower than the 5% Kolmogorov bound: this illustrates the well-known result according to which the estimated reflection coefficient p.d.f. tends to the Gaussian one when parameter σ^2 tends to zero. The Kullback divergence and the Bhattacharyya distance give similar results (Figs. 3 and 4). Similar results may be obtained for higher orders as can been seen in Fig. 5.

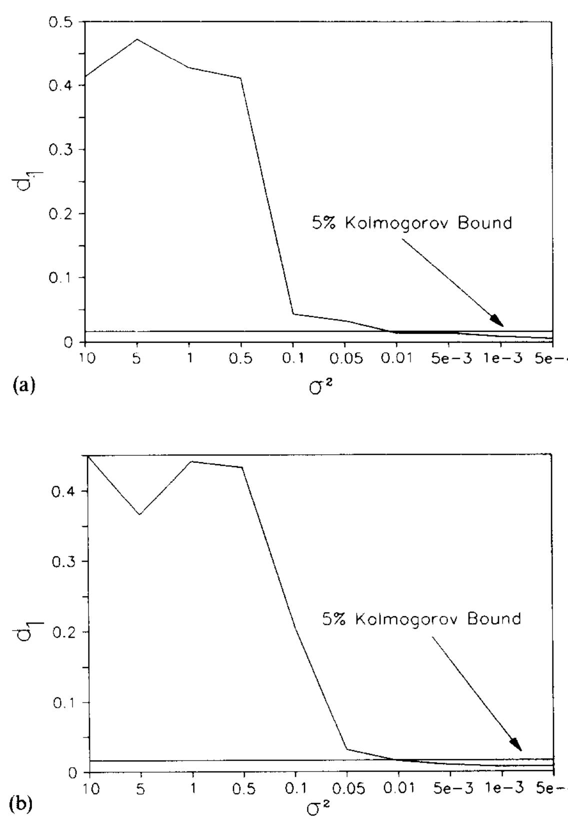


Fig. 2. Kolmogorov distance between estimated reflection coefficient p.d.f. and the normal law for different σ^2 values: (a) Model 1, (b) Model 2.

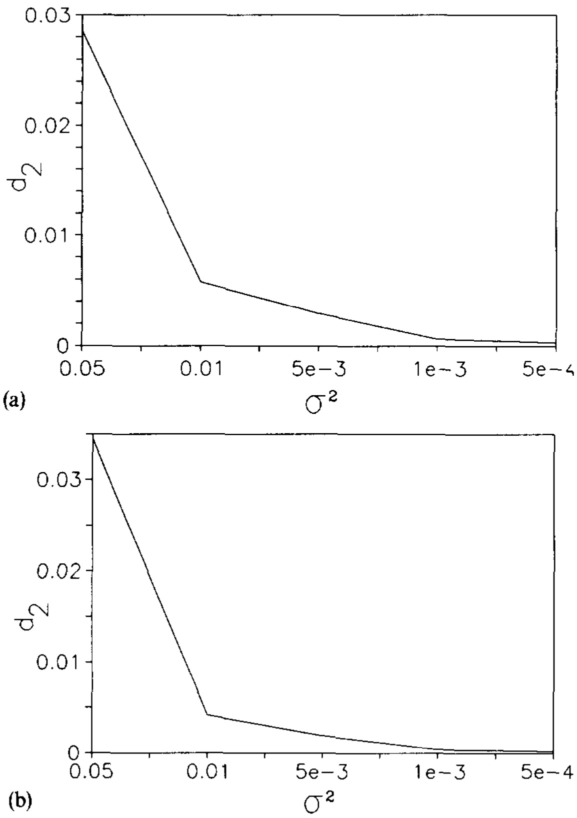


Fig. 3. Kullback divergence between exact reflection coefficient p.d.f. and the normal law for different σ^2 values: (a) Model 1, (b) Model 2.

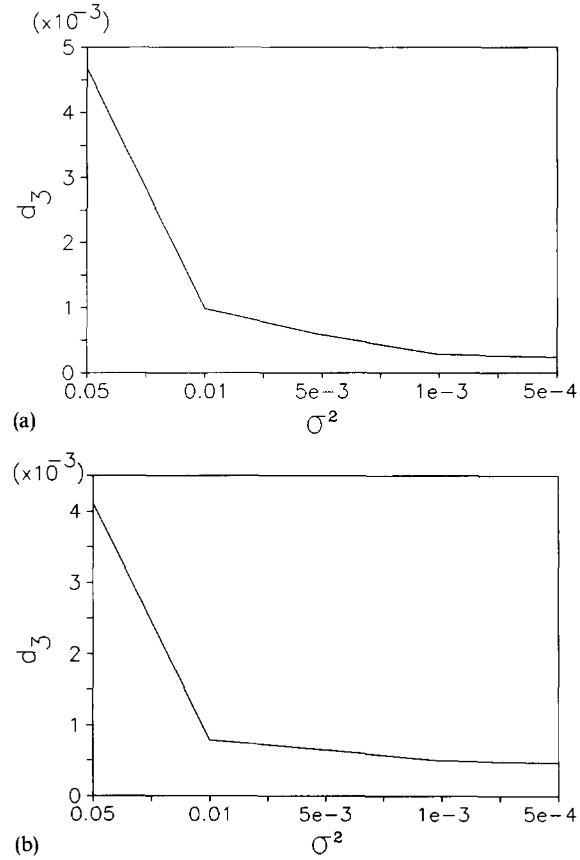
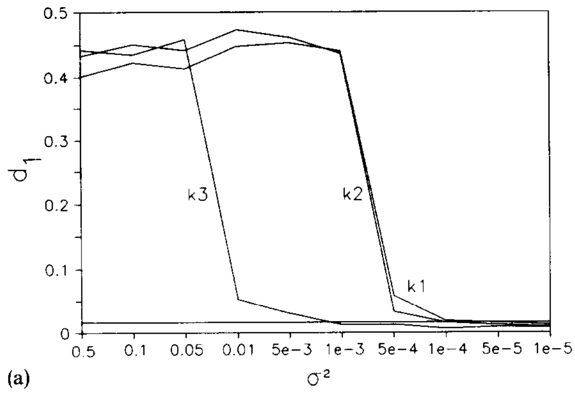


Fig. 4. Bhattacharyya distances between exact reflection coefficient p.d.f. and the normal law for different σ^2 values: (a) Model 1, (b) Model 2.

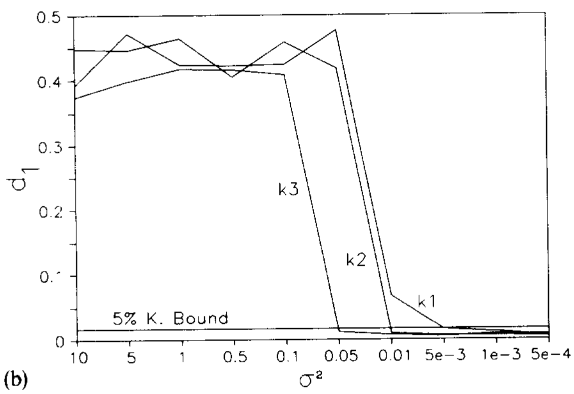
2.4. *Estimated reflection coefficient p.d.f. convexity*

The estimated reflection coefficient p.d.f. may be used in pattern recognition to determine the point clusters in the $\{\hat{k}_i\}$ representation space [14]. Several authors among whom Poublan and Castanié [11] have shown that reflection or cepstrum coefficients, used in pattern recognition and classification with the k-Nearest-Neighbour method (k-NN), give far better results than classification with the centroid distance rule. As far as we know, no theoretical work has been done to explain this. In this part, we propose an analysis of the estimated reflection coefficient p.d.f. convexity which reveals why *the k-NN rule is in many cases a necessary tool in this particular representation space.* It is well known that surprising results may be obtained when classifying with parameters the p.d.f. of which is not convex. In Fig. 6, all points belonging to the second class highlighted with a star are nearer to the first class centroid than to the second class one. So, these points will all be mis-classified with the centroid distance rule. The aim of this part is to study the convexity of the D_K domain defined by $D_K = \{(x_1, \ldots, x_p)/f_1(x_p, \ldots, x_1) = K\}$ (K being a constant) in the case of Gaussian AR parameters. It can be shown recursively that the D_K domain is defined by the following equation:

$$A(x_2, \ldots, x_p)x_1^2 + B(x_2, \ldots, x_p)x_1 + C(x_2, \ldots, x_p) = K', \quad (6)$$



(a)



(b)

Fig. 5. Kolmogorov distance between estimated reflection coefficient p.d.f. and the normal law: (a) poles close to the unit circle, (b) poles close to the origin.

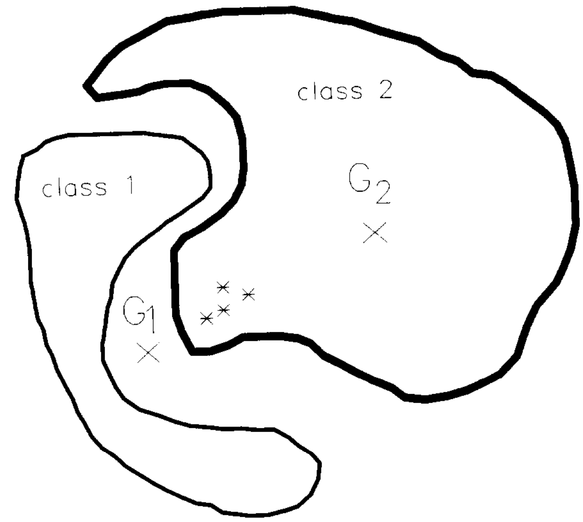


Fig. 6. Classification model.

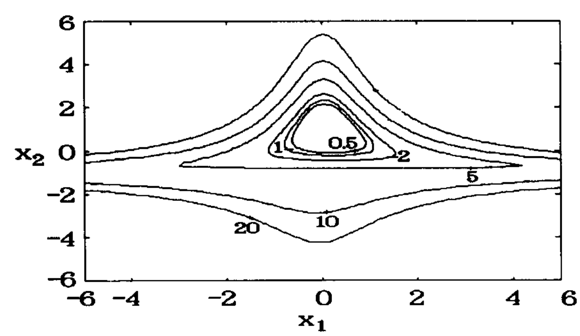


Fig. 7. D_2 domains for Gaussian AR parameters with $a_1 = 0.2$, $a_2 = 0.5$ means and $\Sigma_{\hat{a}} = I$ covariance matrix.

which is a second degree equation with respect to the variable x_1. $A(x_2, \ldots, x_p)$, $B(x_2, \ldots, x_p)$ and $C(x_2, \ldots, x_p)$ can be determined recursively (see Appendix B for order 2). This last equation yields an expression of variable x_1 with respect to $x_2, \ldots, x_p$, K'. A necessary and sufficient condition for the domain D_K to be convex is that the hessian of x_1 with respect to variables $x_2, \ldots, x_p$ is positive definite. This property may be studied numerically but, it fails to show the importance of non-convexity and, in addition, it involves high computational costs. For these reasons, we have preferred to study the reflection coefficient p.d.f. convexity by drawing D_K level curves. For the case of an order 2 AR model with parameters $a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.5$ and covariance matrix $\Sigma_{\hat{a}} = I$, the results obtained are shown in Fig. 7. As can be seen, the D_2 domain is possibly not convex yielding a case in which the centroid distance rule may result in a failure.

3. Cepstrum coefficients

The power cepstrum of a signal $y(n)$ is defined as the inverse Fourier transform of its power spectrum logarithm. Let us consider a zero mean AR process $y(n)$ defined by the following equation:

$$\sum_{k=0}^{p} a_k y(n-k) = v(n) \quad (a_0 = 1,\ a_p \neq 0), \tag{7}$$

$v(n)$ and $\{a_k\}$ being respectively the input white noise and the AR parameters. The cepstrum coefficients of the signal $y(n)$, denoted by c_k, satisfy the following equation:

$$\sum_{k=-\infty}^{+\infty} c_k e^{-j2\pi kf} = \ln\left(\frac{\sigma_v^2}{|A(e^{j2\pi f})|^2}\right), \tag{8}$$

σ_v^2 being the input noise variance and $A(z) = 1 + \sum_{k=1}^{p} a_k z^{-k}$, the AR parameter Z transform. This relation leads to [9]

$$k = 2, \dots, p, \quad kc_k = -ka_k - \sum_{i=1}^{k-1} ic_i a_{k-1}, \tag{9}$$

$$k > p, \quad kc_k = -\sum_{i=1}^{p} (k-i)c_{k-i}a_i, \tag{10}$$

and $c_1 = -a_1$. The estimated cepstrum coefficients $\hat{c}_k$ of an order p AR process are then usually computed from AR parameter estimations $\hat{a}_k$ (obtained with the autocorrelation or covariance method of linear prediction [6]) with relations (9) and (10). It is well known that there exists a one-to-one transformation between the first p AR parameters and the first p cepstrum coefficients [9]. Then all information contained in AR parameters is in the first p cepstrum coefficients and thus only $\hat{c}_1, \dots, \hat{c}_p$ are usually considered.

3.1. Estimated cepstrum coefficient statistics

Using relations (9) and (10), it is possible to show recursively that estimated AR and cepstrum coefficients are linked to the following non-linear relations:

$$\hat{c}_k = -\hat{a}_k + h_{k-1}(\hat{a}_1, \dots, \hat{a}_{k-1}), \tag{11}$$

$$\hat{a}_k = -\hat{c}_k + g_{k-1}(\hat{c}_1, \dots, \hat{c}_{k-1}), \tag{12}$$

h_{k-1} and g_{k-1} being two polynomial functions with $k-1$ variables and constant coefficients (see Appendix C). The following relations may then be derived:

$$\frac{\partial \hat{a}_k}{\partial \hat{c}_k} = -1, \quad \frac{\partial \hat{a}_k}{\partial \hat{c}_i} = 0, \quad i > k.$$

Thus, the Jacobian matrix $J = [\partial \hat{a}_k / \partial \hat{c}_k]$ of the transformation between estimated AR and cepstrum coefficients is lower triangular and all diagonal coefficients are equal to -1. In these conditions, the absolute value of the determinant of the Jacobian matrix is equal to $+1$. If $f_a(x_1, \dots, x_p)$ and $f_c(x_1, \dots, x_p)$ respectively denote the $[\hat{a}_1, \dots, \hat{a}_p]^T$ and $[\hat{c}_1, \dots, \hat{c}_p]^T$ vector p.d.f., the following relation may then be obtained:

$$f_c(x_1, \dots, x_p) = f_a[-x_1, \; -x_2 + g_1(x_1), \dots, \; -x_p + g_{p-1}(x_1, \dots, x_{p-1})]. \tag{13}$$

Polynomial functions g_{k-1} may be determined for any order from relation (9) (see details in Appendix C for orders 2 and 3). The $\hat{c}_k$ p.d.f. may then be determined from Eq. (13) with numerical integration with respect to variables $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_p$.

3.2. The Gaussian case

As described previously (Section 2.2), there are two main applications (i.e. estimation and pattern recognition) for which the estimated AR parameter vector $[\hat{a}_1, \dots, \hat{a}_p]$ is assumed to be Gaussian. The aim of this part is to study the estimated cepstrum parameter p.d.f. in this particular case. We then obtain

$$f_a(x_1, \dots, x_p) = \frac{1}{\sqrt{(2\pi)^p \det(\Sigma_{\hat{a}})}} \times \exp\left[-\frac{1}{2}(x-a)^T \Sigma_{\hat{a}}^{-1}(x-a)\right], \tag{14}$$

with $x = [x_1, x_2, \dots, x_p]^T$, $a = [a_1, \dots, a_p]^T$ and $\Sigma_{\hat{a}}$ being the estimated AR parameter covariance matrix. The estimated cepstrum coefficient p.d.f. is then given from Eq. (13), that is to say,

$$f_c(x_1, x_2, \dots, x_p) = \frac{1}{\sqrt{(2\pi)^p \det(\Sigma_{\hat{a}})}} \times \exp\left[-\frac{1}{2}(\tilde{x}-a)^T \Sigma_{\hat{a}}^{-1}(\tilde{x}-a)\right] \tag{15}$$

with $\tilde{x} = [-x_1, \; -x_2 + g_1(x_1), \dots, \; -x_p + g_{p-1}(x_1, \dots, x_{p-1})]^T$. Obviously, $\{g_k\}_{k=1,\dots,p-1}$ being

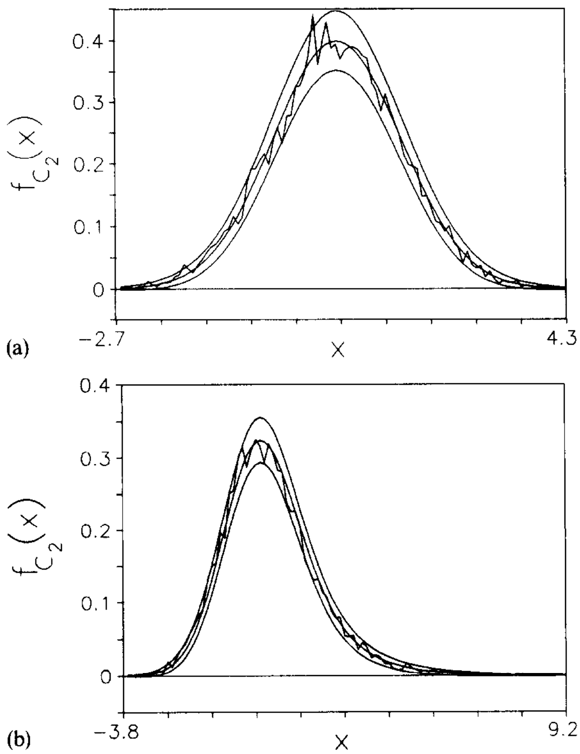


Fig. 8. Estimated and theoretical cepstrum coefficient p.d.f. with 95% confidence intervals: (a) coefficient $\hat{c}_1$, (b) coefficient $\hat{c}_2$.

non-linear functions, *the cepstrum coefficient vector* $\hat{c} = [\hat{c}_1, \dots, \hat{c}_p]^{\mathrm{T}}$ *is non-Gaussian.* In order to illustrate this property, let us consider the previous model with $\sigma^2 = 1$. The results obtained are shown in Fig. 8.

As can be seen from Fig. 8, there is good agreement between the estimated and theoretical cepstrum coefficient p.d.f. These p.d.f. confirm that the first cepstrum coefficient defined by $\hat{c}_1 = -\hat{a}_1$ is Gaussian, whereas the other one is not (asymmetric p.d.f.).

3.3. *Distance to the Gaussian law*

The Gaussian or non-Gaussian nature of estimated cepstrum coefficients is not always so clear as it appears in Figs. 8(a) and 8(b). The aim of this part is to study the distance between the estimated cepstrum coefficient p.d.f. and the Gaussian one, in the case of Gaussian AR parameters with covariance matrix $\Sigma_{\hat{a}} = \sigma^2 I$, as a function of parameter σ^2. This study illustrates the asymptotic normality of estimated cepstrum coefficient (the proof is the same as in [7] for reflection coefficients). The Kullback divergence, Kolmogorov and Bhattacharyya distances between cepstrum coefficient p.d.f. and the normal one have been computed for the two previous models (Section 2.3). The first estimated cepstrum coefficient (which is the opposite of the first AR parameter) being Gaussian, only the second one has been considered (Figs. 9–11).

As for reflection coefficients, the distance between the estimated cepstrum coefficient p.d.f. and the Gaussian one decreases with parameter σ^2. This

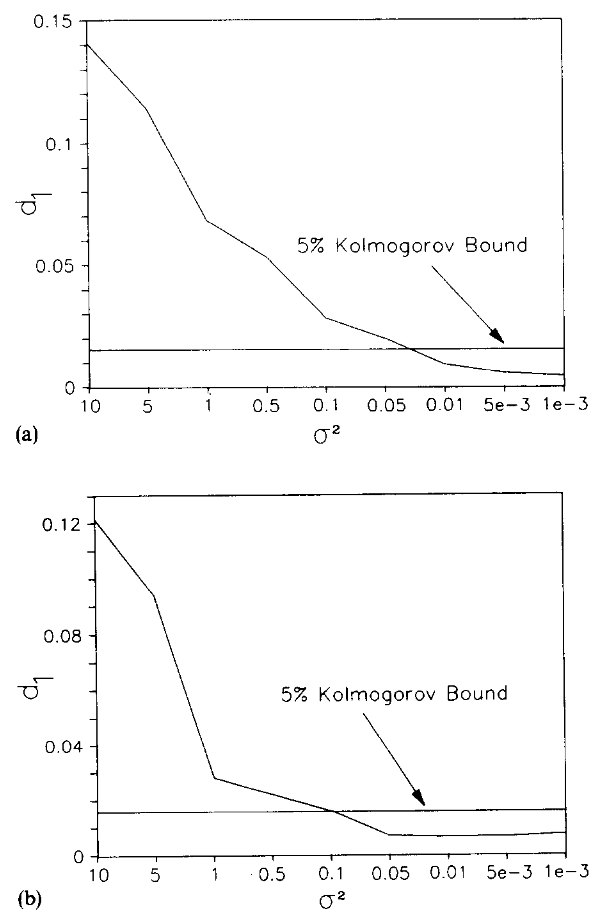


Fig. 9. Kolmogorov distance between second estimated cepstrum coefficient p.d.f. and the normal law for different σ^2 values: (a) Model 1, (b) Model 2.

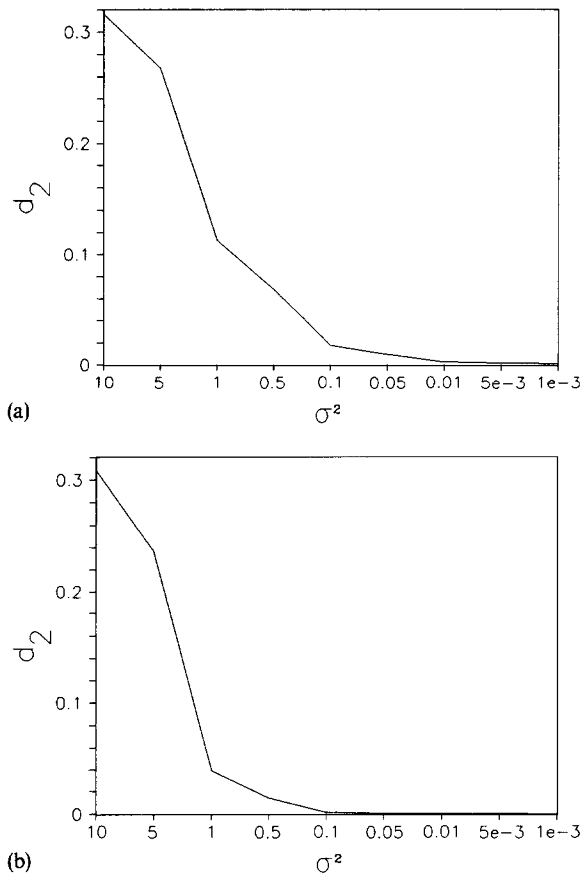


Fig. 10. Kullback divergence between exact cepstrum coefficient p.d.f. and the normal law for different σ^2 values: (a) Model 1, (b) Model 2.

illustrates the asymptotic normality of estimated cepstrum coefficients [7]. The same results may be obtained for higher order models (Fig. 12).

3.4. Estimated cepstrum coefficient p.d.f. convexity

The aim of this part is to study the convexity of the domain D_K defined by

$$D_K = \{(x_1, \dots, x_p)/f_c(x_1, \dots, x_p) = K\}.$$

In the case of Gaussian AR parameters, according to Eq. (15), we obtain

$$D_K = \{(x_1, \dots, x_p)/(\tilde{x} - a)^{\mathrm{T}} \Sigma_{\hat{a}}^{-1}(\tilde{x} - a) = K'\}, \quad (16)$$

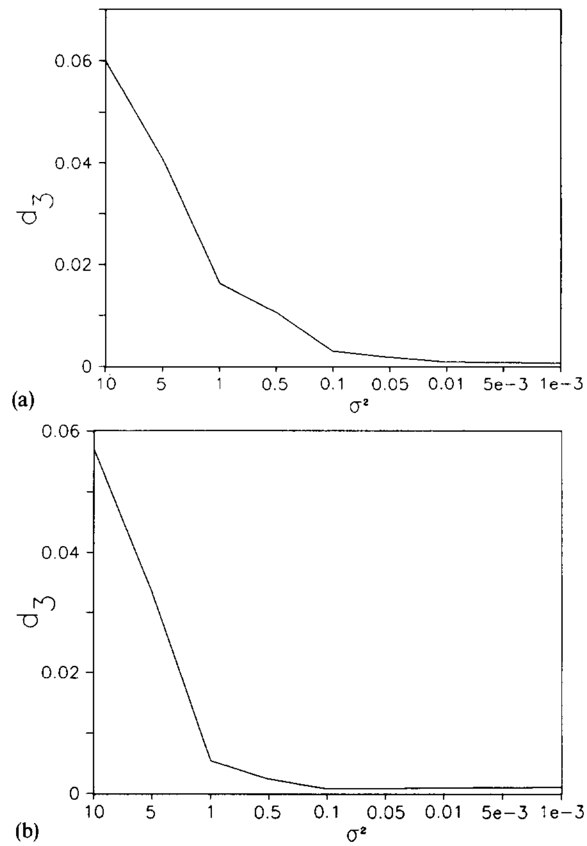


Fig. 11. Bhattacharyya distances between exact cepstrum coefficient p.d.f. and the normal law for different σ^2 values: (a) Model 1, (b) Model 2.

with $K' = -2\ln[\sqrt{K^2(2\pi)^p \det(\Sigma_{\hat{a}})}]$. $(\tilde{x} - a)^{\mathrm{T}} \times \Sigma_{\hat{a}}^{-1}(\tilde{x} - a) = K'$ is a 2nd degree equation with respect to the unknown x_p, the coefficients of which depend on $x_1, \dots, x_{p-1}$. The equation yields

$$x_p = B(x_1, \dots, x_{p-1}) \pm \sqrt{K'' + B^2(x_1, \dots, x_{p-1}) - A(x_1, \dots, x_{p-1})}, \quad (17)$$

K'' being a constant depending on K', $A(x_1, \dots, x_{p-1})$ and $B(x_1, \dots, x_{p-1})$ being two functions depending on variables $x_1, \dots, x_{p-1}$ (given in Appendix D for orders 2 and 3). In the case of the previous order 2 AR model ($a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.5$ and an order 3 AR model defined by $a_0 = 1, a_1 = 0.5, a_2 = 0.2, a_3 = 0.5$ with covariance matrix $\Sigma_{\hat{a}} = I$, the results obtained are illustrated in Figs. 13 and 14.

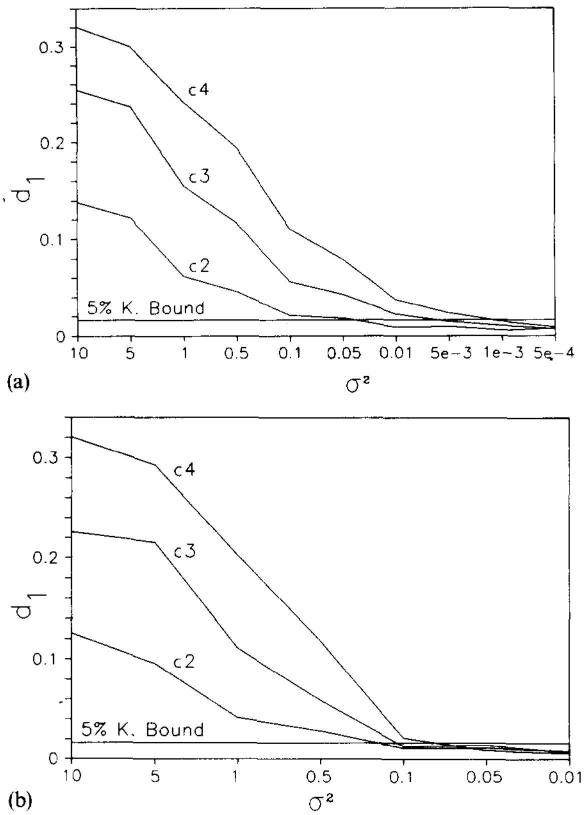


Fig. 12. Kolmogorov distance between estimated cepstrum coefficient p.d.f. and the normal law for an order 4 AR model and for different σ^2 values: (a) poles close to the unit circle, (b) poles close to origin.

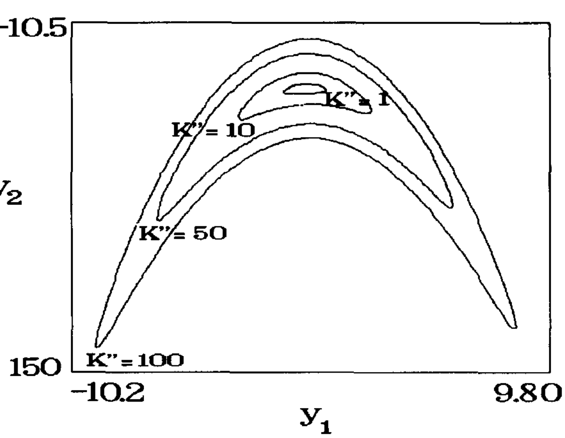


Fig. 13. D_2 level curves.

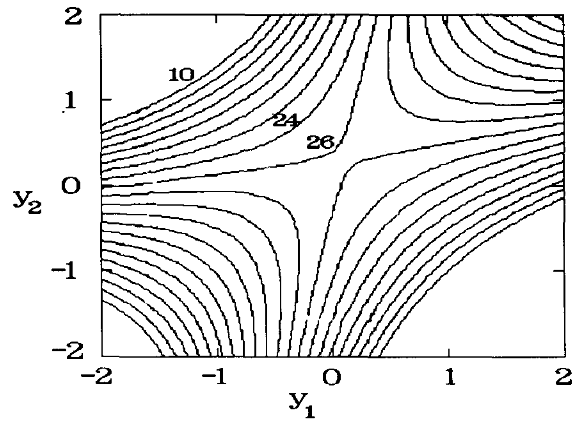


Fig. 14. D_3 level curves.

The results show that the $\{\hat{c}_k\}$ p.d.f. may not be convex, yielding a case in which the centroid distance classifier may result in a failure.

4. Conclusion

The first result of this paper is a recursive means of determining the exact p.d.f. of estimated cepstrum and reflection coefficients from that of estimated AR parameters. Knowledge of the latter may be justified mainly in the following two applications:
- in estimation theory, according to the Mann and Wald theorem, estimated AR parameters may be assumed to be Gaussian when they are estimated with a sufficient signal point number and
- in pattern recognition, the AR parameter p.d.f. which characterises within-class scattering is usually assumed to be Gaussian.

Our second step is to study the estimated cepstrum and reflection p.d.f. in the case of Gaussian estimated AR parameters. These p.d.f. are clearly non-Gaussian but may be very close to the Gaussian law. Simulation results are proposed for two particular models giving some distances between the estimated cepstrum or reflection coefficient p.d.f. and the Gaussian law.

In the last part of the paper, estimated cepstrum and reflection coefficient p.d.f. convexity are studied. Several examples show that these p.d.f. are possibly non-convex which may raise problems when classifying with the centroid distance rule.

Appendix A. Study of the transformation between V_i and V_{i-1}.

Let us denote $v_i(j)$ with $j = 1, \dots, p$ components of the vector $\hat{V}_i$ such that

$\hat{V}_i^T = [v_i(1), \dots, v_i(p)]$.

The $\hat{V}_i$ and $\hat{V}_{i-1}$ components are linked by the following relations:

$$v_{i-1}(j) = v_i(j) \quad \text{for } j = 1, \dots, p-i+1, \tag{A.1}$$

$$v_{i-1}(j) = \hat{a}_{p-j+1}^{(i-1)} = \frac{v_i(j) - v_i(p-i+1)v_i(2p+2-i-j)}{1 - v_i^2(p-i+1)} \quad \text{for } j = p-i+2, \dots, p. \tag{A.2}$$

Using relation $v_i(p-i+1) = \hat{a}_i^{(i)} = v_{i-1}(p-i+1)$, we get

$$v_{i-1}(j) = \hat{a}_{p-j+1}^{(i-1)} = \frac{v_i(j) - v_{i-1}(p-i+1)v_i(2p+2-i-j)}{1 - v_{i-1}^2(p-i+1)} \quad \text{for } j = p-i+2, \dots, p, \tag{A.3}$$

which finally leads to

$$v_i(j) - v_{i-1}(p-i+1)v_i(2p+2-i-j) = [1 - v_{i-1}^2(p-i+1)]v_{i-1}(j) \quad \text{for } j = p-i+2, \dots, p. \tag{A.4}$$

Two cases have to be considered according to the odd or evenness of integer i. Namely, $2p+2-i-j$ and j are equal for $i = 2(p+1-j)$ and this may happen only if integer i is even.

Case 1: *i odd.* $2p+2-i-j$ and j are different whatever integer j and belong to the set $\{p-i+2, \dots, p\} = \{p-i+1+1, \dots, p-i+1+i-1\}$. The $i-1$ equations (A.4) with $i-1$ unknown may be rewritten as $(i-1)/2$ systems of two equations with two unknowns:

$$v_i(k) - v_{i-1}(p-i+1)v_i(2p+2-i-k) = [1 - v_{i-1}^2(p-i+1)]v_{i-1}(k),$$
$$v_i(2p+2-i-k) - v_{i-1}(p-i+1)v_i(k) = [1 - v_{i-1}^2(p-i+1)]v_{i-1}(2p+2-i-k), \tag{A.5}$$

k varying from $p+2-i$ to $p-(i-1)/2$. The order k system solution is given by

$$v_i(2p+2-i-k) = v_{i-1}(p-i+1)v_{i-1}(k) + v_{i-1}(2p+2-i-k),$$
$$v_i(k) = v_{i-1}(k) + v_{i-1}(2p+2-i-k) \times v_{i-1}(p-i+1). \tag{A.6}$$

The Jacobian corresponding to this one-to-one transformation between $\hat{V}_i$ and $\hat{V}_{i-1}$ is the block matrix:

$$M = \begin{bmatrix} A & B \\ B & D \end{bmatrix},$$

where A is the identity matrix of order $p-i+1$,

$$B = \begin{bmatrix} 0 & \cdots & 0 \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ 0 & \cdots & 0 \\ v_{i-1}(p) & \cdots & v_{i-1}(p+2-i) \end{bmatrix}$$

for ($p-i+1$ rows and $i-1$ columns),

C being the null matrix ($i-1$ rows and $p-i+1$ columns) and

$$\begin{bmatrix} 1 & 0 & 0 & . & . & 0 & 0 & a \\ 0 & 1 & 0 & . & . & 0 & a & 0 \\ 0 & 0 & 1 & . & . & a & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & a & . & . & 1 & 0 & 0 \\ 0 & a & 0 & . & . & 0 & 1 & 0 \\ a & 0 & 0 & . & . & 0 & 0 & 1 \end{bmatrix}$$

is an $i-1$ square matrix,

with $a = v_{i-1}(p-i+1)$.

The determinant of the Jacobian matrix may then be computed:

$$\det(M) = [1 - a^2]^{(i-1)/2}, \quad i > 1. \tag{A.7}$$

Case 2: *i even*. $2p+2-i-j$ and j are equal for $i = 2(p+1-j)$ that is to say for $j = (2p+2-i)/2$. The $i-1$ equations (A.4) with $i-1$ unknowns may then be rewritten as $(i-2)/2$ two equation and two unknown systems plus one equation with one unknown:

$$v_i(k) - v_{i-1}(p-i+1)v_i(2p+2-i-k) = [1 - v_{i-1}^2(p-i+1)]v_{i-1}(k), \tag{A.8}$$

$$v_i(2p+2-i-k) - v_{i-1}(p-i+1)v_i(k) = [1 - v_{i-1}^2(p-i+1)]v_{i-1}(2p+2-i-k),$$

k varying from $p+2-i$ to $p-(i-2)/2$ and

$$v_i\left(\frac{2p-i+2}{2}\right) = [1 + v_{i-1}(p-i+1)]v_{i-1}\left(\frac{2p-i+2}{2}\right). \tag{A.9}$$

The last equation (A.9) allows us to determine the variable $v_i((2p-i+2)/2)$. Variables $v_i(k)$ and $v_i(2p+2-i-\mathrm{k})$ may be determined as a function of $v_{i-1}(p-i+1)$, $v_{i-1}(k)$ and $v_{i-1}(2p+2-i-k)$ from system (A.8). The Jacobian corresponding to this one-to-one transformation between V_i and V_{i-1} is the block matrix:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

A, B and C being defined as in the previous case. Only the matrix D is different:

$$D = \begin{bmatrix} 1 & 0 & 0 & . & . & . & . & . & 0 & 0 & a \\ 0 & 1 & 0 & . & . & . & . & . & 0 & a & 0 \\ 0 & 0 & 1 & . & . & . & . & . & a & 0 & 0 \\ . & . & . & . & 1 & 0 & a & . & . & . & . \\ . & . & . & . & 0 & 1+a & 0 & . & . & . & . \\ . & . & . & . & a & 0 & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & a & . & . & . & . & . & 1 & 0 & 0 \\ 0 & a & 0 & . & . & . & . & . & 0 & 1 & 0 \\ a & 0 & 0 & . & . & . & . & . & 0 & 0 & 1 \end{bmatrix}$$

with $a = v_{i-1}(p-i+1)$. The determinant of this Jacobian matrix may then be computed.

$$\det(M) = (1+a)[1-a^2]^{(i-2)/2}, \quad i > 0. \tag{A.10}$$

The study of these two cases leads to the determination of the V_{i-1} p.d.f. as a function of that of V_i. The following results are obtained:
(i) when i is odd:

$$f_{i-1}(x_p, \ldots, x_1) = (1 - x_i^2)^{(i-1)/2} \times f_i(x_p, \ldots, x_i, x_{i-1} + x_i x_1, \ldots, x_1 + x_i x_{i-1});$$

(ii) when i is even:

$$f_{i-1}(x_p, \ldots, x_1) = (1 + x_i)(1 - x_i^2)^{(i-2)/2} f_i(x_p, \ldots, x_i, x_{i-1} + x_i x_1, \ldots, (1+x_i)x_{i/2}, \ldots, x_1 + x_i x_{i-1}).$$

Appendix B. Expression of $A(x_2)$, $B(x_2)$ and $C(x_2)$ for an order 2 AR model

The two first reflection coefficients are linked to AR parameters by the following commonly used relations:

$$k_1 = \frac{a_1}{1+a_2}, \qquad k_2 = a_2.$$

In the Gaussian case, estimated AR parameter p.d.f. is given by

$$f_2(x_1, x_2) = \left[2\pi\sqrt{\det(\Sigma_{\hat{a}})}\right]^{-1} \times \exp\left[-\frac{1}{2}(x-a)^{\mathrm{T}} \Sigma_{\hat{a}}^{-1}(x-a)\right],$$

with $x^{\mathrm{T}} = (x_1, x_2)$, $a^{\mathrm{T}} = (a_1, a_2)$ and $\Sigma_{\hat{a}}$ being the estimated AR parameter covariance matrix. Reflection coefficient p.d.f. is then given by

$$f_1(x_1, x_2) = \left[2\pi\sqrt{\det(\Sigma_{\hat{a}})}\right]^{-1} \exp\left[-\frac{1}{2}(\tilde{x}-a)^{\mathrm{T}} \Sigma_{\hat{a}}^{-1}(\tilde{x}-a)\right] \times |1 + x_2|,$$

with $\tilde{x}^{\mathrm{T}} = [x_1(1+x_2), x_2]$. D_2 domains may then be determined theoretically. The simplified case $\Sigma_{\hat{a}}^{-1} = \Sigma_{\hat{a}} = I$ yields:

$$[x_1(1+x_2) - a_1)]^2 + (x_2 - a_2)^2 - 2\ln|1+x_2| = K',$$

hence

$$A(x_2) = (1+x_2)^2,$$

$$B(x_2) = -2a_1(1+x_2),$$

$$C(x_2) = a_1^2 + (x_2 - a_2)^2 - 2\ln|1+x_2| - K'.$$

Appendix C. Study of polynomial relations between AR and cepstrum coefficients

The aim of this appendix is to prove that AR and cepstrum coefficients are related for $1 < k \leqslant p$ with the following non-linear relations:

$$c_k = -a_k + h_{k-1}(a_1, \ldots, a_{k-1}),$$

h_{k-1} being a polynomial with $k-1$ variables and constant coefficients. The demonstration is done recurrently, taking into account the following relations:

$$k = 2, \ldots, p, \quad kc_k = -ka_k - \sum_{i=1}^{k-1} ic_i a_{k-i}.$$

The result is true for $k = 2$.

The second cepstrum coefficient is such that $c_2 = -a_2 + \frac{1}{2}a_1^2$ and hence $h_1(a_1) = \frac{1}{2}a_1^2$. *Let us assume that the result is true for $i \leqslant k-1$ and let us prove that it is true for order k.* Following recurrence hypothesis, for every $i = 2, \ldots, k-1$, $ic_i a_{k-i}$ is a polynomial function depending on variables $a_1, \ldots, a_i$ and a_{k-i} with $k - i \in \{1, \ldots, k-1\}$. Thus, $-\sum_{i=1}^{k-1} \frac{i}{k} c_i a_{k-i}$ is a polynomial function with variables $a_1, \ldots, a_{k-1}$. The order k cepstrum coefficient being defined by $c_k = -a_k - \sum_{i=1}^{k-1} \frac{i}{k} c_i a_{k-i}$, the result is true for $i = k$.

A similar recurrence leads to following relations:

$$a_k = -c_k + g_{k-1}(c_1, \ldots, c_{k-1}).$$

For orders 2 and 3, following results are obtained:

Order 2:

$$c_1 = -a_1, \quad c_2 = -a_2 + \frac{1}{2}a_1^2,$$

$$a_1 = -c_1, \quad a_2 = -c_2 + \frac{1}{2}c_1^2.$$

Hence,

$$h_1(a_1) = \frac{1}{2}a_1^2 \quad \text{and} \quad g_1(c_1) = \frac{1}{2}c_1^2.$$

The vector $\tilde{x}^{\mathrm{T}} = [-x_1, -x_2 + g_1(x_1), \ldots, -x_n + g_{n-1}(x_1, \ldots, x_{n-1})]$ is then defined by

$$\tilde{x} = \begin{bmatrix} -x_1 \\ -x_2 + \frac{1}{2}x_1^2 \end{bmatrix}.$$

Order 3:

$$c_1 = -a_1, \quad c_2 = -a_2 + \frac{1}{2}a_1^2,$$

$$c_3 = -a_3 + a_1 a_2 - \frac{1}{3}a_1^3,$$

$$a_1 = -c_1, \quad a_2 = -c_2 + \frac{1}{2}c_1^2.$$

$$a_3 = -c_3 + c_1 c_2 - \frac{1}{6}c_1^3.$$

Hence,

$$h_2(a_1, a_2) = a_1 a_2 - \frac{1}{3}a_1^3 \quad \text{and}$$

$$g_2(c_1, c_2) = c_1 c_2 \frac{1}{6}c_1^3.$$

$\tilde{x}$ is then given by

$$\tilde{x} = \begin{bmatrix} -x_1 \\ -x_2 + \frac{1}{2}x_1^2 \\ -x_3 + x_1 x_2 - \frac{1}{6}x_1^3 \end{bmatrix}.$$

Appendix D. Determination of functions $A(x_1)$, $B(x_1)$, $A(x_1,x_2)$, $B(x_1,x_2)$

Let us denote by $\Sigma_{\hat{a}}^{-1} = (\alpha_{ij})$, the inverse AR parameter covariance matrix. Resolution of Eq. (16) with respect to x_p, p being the AR model order, gives the following results:

Order 2:

$$A(x_1) = \frac{1}{\alpha_{22}}\left[\alpha_{11}(x_1 + a_1)^2 - 2\alpha_{12}(x_1 + a_1) \times \left(\frac{1}{2}x_1^2 - a_2\right) + \alpha_{22}\left(\frac{1}{2}x_1^2 - a_2\right)^2\right],$$

$$B(x_1) = \left(\frac{1}{2}x_1^2 - a_2\right) - \frac{\alpha_{12}}{\alpha_{22}}(x_1 + a_1).$$

Order 3:

$$B(x_1,x_2) = \frac{1}{2\alpha_{33}}\left[-\alpha_{13}(x_1 + a_1) - \alpha_{23}\left(-x_2 + \frac{1}{2}x_1^2 - a_2\right) - \alpha_{33}\left(x_1x_2 - \frac{1}{6}x_1^3 - a_3\right)\right]$$

$$A(x_1,x_2) = \frac{1}{\alpha_{33}}\left[\alpha_{11}(x_1 + a_1)^2 + \alpha_{22}\left(-x_2 + \frac{1}{2}x_1^2 - a_2\right)^2 + \alpha_{33}\left(x_1x_2 - \frac{1}{6}x_1^3 - a_3\right)^2 - 2\alpha_{12}(x_1 + a_1)\left(-x_2 + \frac{1}{2}x_1^2 - a_2\right) - 2\alpha_{13}(x_1 + a_1)\left(x_1x_2 - \frac{1}{6}x_1^3 - a_3\right) + 2\alpha_{33}\left(-x_2 + \frac{1}{2}x_1^2 - a_2\right) \times \left(x_1x_2 - \frac{1}{6}x_1^3 - a_3\right)^2\right].$$

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